BLG 354E Signals & Systems

03.05.2021

Week 10

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Before $h[n]$: impulse response of DT systems $\quad x(n) \rightarrow \boxed{h[n]} \rightarrow y(n)$

Both characterize the system

$H(e^{j\hat{w}})$: frequency response " " $\rightarrow H(e^{j\hat{w}}) = |H(e^{j\hat{w}})| \cdot e^{j\measuredangle H(e^{j\hat{w}})}$

$\boxed{e^{j\hat{w}_0 n}} \rightarrow \boxed{H(e^{j\hat{w}})} \rightarrow H(e^{j\hat{w}_0}) \boxed{e^{j\hat{w}_0 n}}$

(complex sinusoid)

(for a real sinusoid) $\boxed{\cos(\hat{w}_0 n)} \rightarrow \boxed{H(e^{j\hat{w}})} \rightarrow |H(e^{j\hat{w}_0})| \boxed{\cos(\hat{w}_0 n + \measuredangle H(e^{j\hat{w}_0}))}$

evaluated at the input frequency

Ex: For FIR filter $h(n) = \{1, 2, 1\}$

Last time exercise

Given: $x[n] = \underbrace{4}_{x_1(n)} + \underbrace{3\cos\left(\boxed{\frac{\pi}{3}}n - \frac{\pi}{2}\right)}_{x_2(n)} + \underbrace{3\cos\left(\boxed{\frac{7\pi}{8}}n\right)}_{x_3(n)}$

0 freq. freq freq.

$\Rightarrow y(n) = \underbrace{16} + (9)\cos\left(\frac{\pi}{3}n - \frac{\pi}{2} - \frac{\pi}{3}\right) + (3) . 0.152 \cos\left(\frac{7\pi}{8}n - \frac{7\pi}{8}\right)$

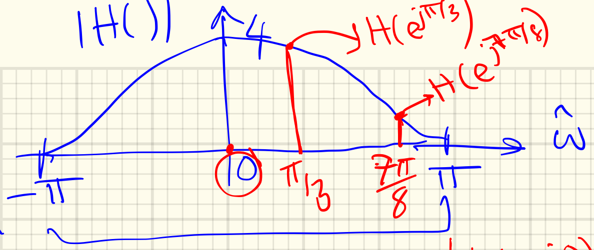
Did you find the soln? Using superposition. →

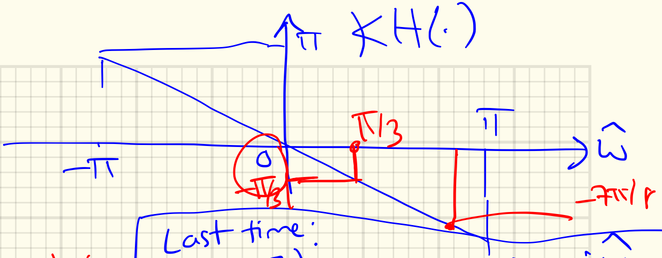
$x_1(n) \rightarrow \boxed{H(\cdot)} \rightarrow y_1(n)$

$x_2(n) \rightarrow \boxed{H(\cdot)} \rightarrow y_2(n)$

$x_3(n) \rightarrow \boxed{H(\cdot)} \rightarrow y_3(n)$

Given FIR filter in freq. domain





Last time:

$$H(e^{j\hat{\omega}}) = (2+2\cos\hat{\omega})e^{-j\hat{\omega}}$$

For $x_1(n)$, $\hat{\omega}_1 = 0$

$$y_1(n) = \underbrace{x_1(n)}_{4} \cdot \underbrace{|H(e^{j0})|}_{4} = 16$$

For $x_2(n)$: $\hat{\omega}_2 = \frac{\pi}{3} \rightarrow H(e^{j\pi/3}) = \underline{3}\, e^{\underline{-j\pi/3}}$

For $x_3(n)$ $\hat{\omega}_3 = \frac{7\pi}{8} \rightarrow H(e^{j7\pi/8}) = \left(2+2\cos\underbrace{\frac{7\pi}{8}}\right) e^{-j7\pi/8}$

$$= 0.152\, e^{-j7\pi/8}$$

→ multiplying mag. due system magnitude response

$$\rightarrow y_2[n] = 3 \cdot \boxed{|3|} \cos\left(\frac{\pi}{3}n - \frac{\pi}{2} \boxed{- \frac{\pi}{3}}\right)$$

→ phase due to the system phase response

$$y_3[n] = 3 \cdot \boxed{(0.152)} \cos\left(\frac{7\pi}{8}n \boxed{- \frac{7\pi}{8}}\right)$$

$$\Rightarrow y(n) = y_1(n) + y_2(n) + y_3(n)$$

★ Better to know which domain (time or frequency) to work with efficiently computationally, depending on your input signal components.

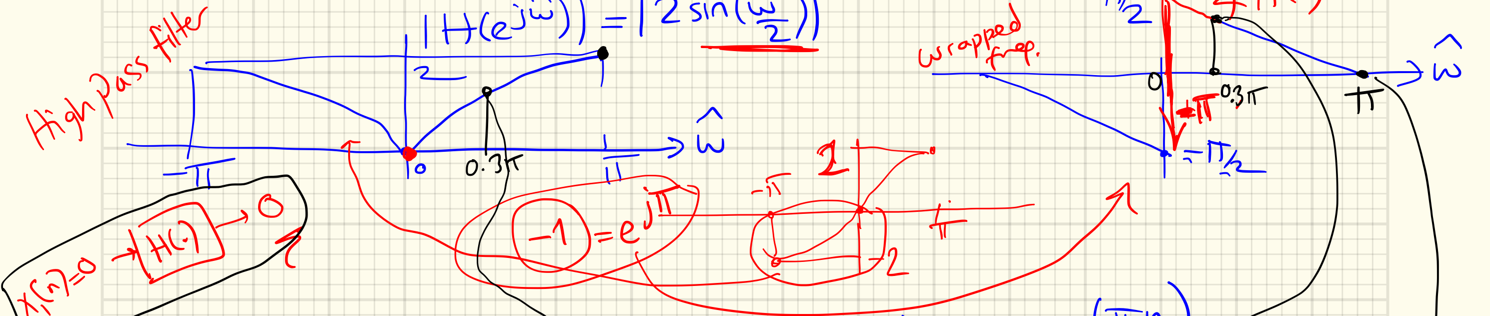
Ex: First difference system: $y[n] = x[n] - x[n-1]$

$h[n] = \delta[n] - \delta[n-1]$

$\angle H(\cdot) = -\frac{\hat{\omega}}{2} + \frac{\pi}{2}$

$$\rightarrow H(e^{j\hat{\omega}}) = \sum_{k=0}^{1} h[k] e^{-j\hat{\omega}k} = \dots = 2\sin\left(\frac{\hat{\omega}}{2}\right) e^{-j\left(\frac{\hat{\omega}}{2} - \frac{\pi}{2}\right)}$$

$$|H(e^{j\hat{\omega}})| = \left|2\sin\left(\frac{\hat{\omega}}{2}\right)\right|$$



exercise: Let $x[n] = \underbrace{4} + 2\cos(\underbrace{0.3\pi}n - \pi/4) + \cos(\underbrace{\pi}n)$

$$y[n] = ? = \cancel{4.0} + 2|H(e^{j0.3\pi})|.\cos\left(0.3\pi n - \frac{\pi}{4} + \angle H(e^{j0.3\pi})\right) + 1.(2)\cos(\pi n + 0)$$

0

Frequency Response of Difference Equation Systems:

General form:

$$\boxed{\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]}$$

We assume $a_0 = 1$, $a_k = 0$ $k \neq 0$. $\Bigg\} \Rightarrow y[n] = \sum_{k=0}^{M} b_k\, x[n-k]$

When $x[n] = e^{j\hat{\omega} n} \rightarrow \boxed{y[n] = H(e^{j\hat{\omega}})\, e^{j\hat{\omega} n}}$

substitute

$$\sum_{k} a_k \boxed{H(e^{j\hat{\omega}})}\, e^{j(n-k)\hat{\omega}} = \sum_{k} b_k\, e^{j\hat{\omega}(n-k)}$$

$$\Rightarrow H(e^{j\hat{\omega}})\, e^{j\hat{\omega} n} \cdot \boxed{\sum_{k} a_k e^{-j\hat{\omega} k}} = \sum_{k} b_k\, e^{-j\hat{\omega} k} \cdot e^{j\hat{\omega} n}$$

$$\Rightarrow \boxed{H(e^{j\hat{\omega}}) = \frac{\sum_{k} b_k\, e^{-j\hat{\omega} k}}{\sum_{k} a_k\, e^{-j\hat{\omega} k}}}$$

for us $a_0 = 1$, $\forall$ other $a_k = 0$

$$\Rightarrow \boxed{H(e^{j\hat{\omega}}) = \sum_{k} b_k\, e^{-j\hat{\omega} k}}$$

$$\text{Ex: } y[n] = \frac{1}{6}\{ x[n] + x[n-1] + \cdots + x[n-5] \}$$ $\boxed{\text{6 pt RAF}}$.

$$h(n) = \{ \underset{n=0}{\underline{\tfrac{1}{6}}}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6} \}$$

$$H(e^{j\hat{\omega}}) = \frac{1}{6}\Big(\underbrace{1 + e^{-j\hat{\omega}}} + \underbrace{e^{-j\hat{\omega}2} + e^{-j\hat{\omega}3}} + \underbrace{e^{-j\hat{\omega}4} + e^{-j\hat{\omega}5}} \Big)$$

$$H(e^{j\hat{\omega}}) = \frac{1}{6}\Big((1+e^{-j\hat{\omega}}) + e^{-j2\hat{\omega}}(1+e^{-j\hat{\omega}}) + e^{-j4\hat{\omega}}(1+e^{-j\hat{\omega}}) \Big)$$

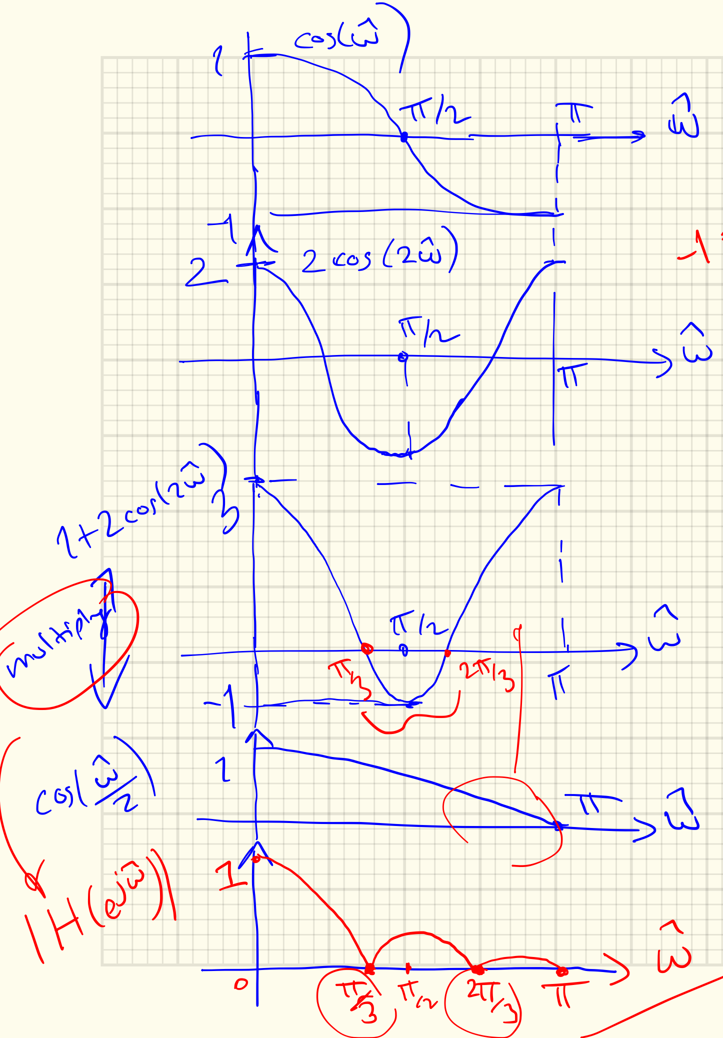
$$= \frac{1}{6}\underbrace{(1+e^{-j\hat{\omega}})}(1 + e^{-j2\hat{\omega}} + e^{-j4\hat{\omega}})$$

$$H(\cdot) = \frac{1}{6}\cdot 2e^{-j\hat{\omega}/2}\cos\left(\frac{\hat{\omega}}{2}\right)\cdot e^{-j\hat{\omega}2}\Big(\underbrace{e^{j\hat{\omega}2} + 1 + e^{-j\hat{\omega}2}}_{1+2\cos(2\hat{\omega})}\Big)$$

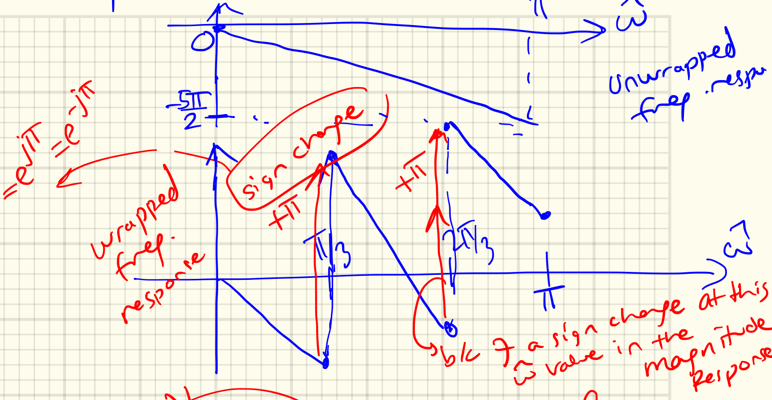
$$H(e^{j\hat{\omega}}) = \frac{1}{3}e^{-j\frac{5}{2}\hat{\omega}}\underbrace{\cos\left(\frac{\hat{\omega}}{2}\right)}_{\text{mag. }(0,1)}\underbrace{(\underbrace{2\cos(2\hat{\omega})}_{\text{in }(-2,2)}+1)} \rightarrow (-1,3)$$

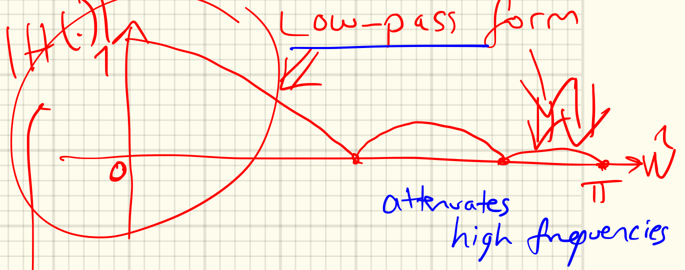
↑ plot this

$$|H(e^{j\hat{\omega}})| = \boxed{\frac{1}{3}}\underbrace{\underbrace{\left|\cos\left(\frac{\hat{\omega}}{2}\right)\cdot(2\cos(2\hat{\omega})+1)\right|}_{\text{mag. }(0,3)}} \rightarrow \text{mag } (0,1)$$



$\measuredangle H(e^{j\hat{\omega}}) = -5\hat{\omega}/2.$





zeros at multiples of 2π/6

$\frac{2\pi}{6}k.$

in general

For a L-pt. RAF

zeros at $\frac{2\pi}{L}k.$

Exercise: Now if we had the system (for comparison)

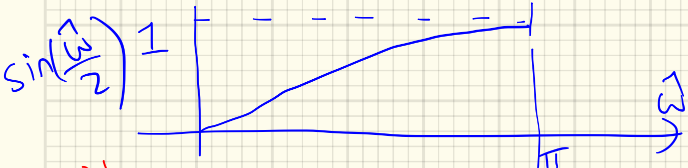
$$y[n] = \frac{1}{6}\left\{\underbrace{x[n]-x[n-1]}+\underbrace{x[n-2]-x[n-3]}+\underbrace{x[n-4]-x[n-5]}\right\}$$

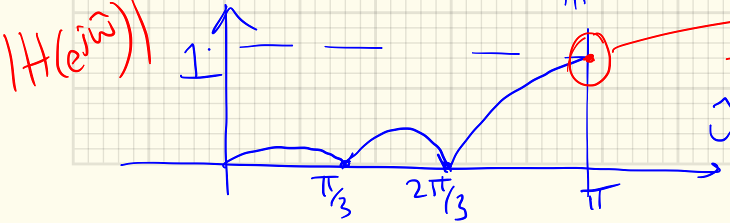
takes 6 most recent inputs & form the 1st differences.

→ this time it would attenuate low freq components in the input

Calculate to find

$$H(e^{j\hat{\omega}}) = \frac{1}{3}\underbrace{e^{j\pi/2}e^{-j\frac{5}{2}\hat{\omega}}}_{\measuredangle H(\cdot)}\sin\left(\frac{\hat{\omega}}{2}\right)\underbrace{\left(2\cos(2\hat{\omega})+1\right)}$$



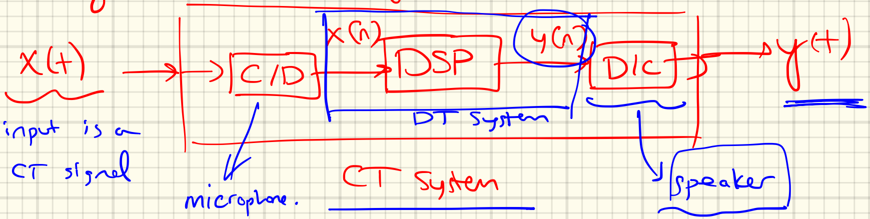


For high freq. close to π,

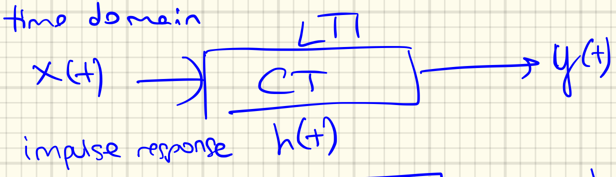
→ high pass: $|H(\cdot)|$ has a gain ≈ 1.

Frequency Response of CT Systems (Chapter 10 SP First)

If the input is a physical CT signal



CT Systems in time domain



If x(t) is a sinusoidal signal → $\boxed{CT}$ → output is again a sinusoidal but modified

$$\underbrace{e^{j\omega_0 t}} \longrightarrow \boxed{H(j\omega)} \longrightarrow \boxed{H(j\omega_0)}\cdot e^{j\omega_0 t}$$

Freq. Response is evaluated at input freq.

$H(j\omega)$: only a function freq variable ω.

Let $\underbrace{x(t) = A\, e^{j\phi}\, e^{j\omega t}} \longrightarrow \boxed{h(t)} \longrightarrow y(t)$

We know: $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)\, \underbrace{x(t-\tau)}\, d\tau$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, \underline{A e^{j\phi}}\, \underline{e^{j\omega(t-\tau)}}\, d\tau = \underbrace{A e^{j\phi} e^{j\omega t}} \underbrace{\int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau}_{\triangleq H(j\omega)}$$

$\triangleq H(j\omega)$: Fourier Transform of $h(t)$.

$$= \underbrace{x(t)}_{A e^{j\phi} e^{j\omega t}} \cdot H(j\omega).$$

: only for sinusoidal.

$$\boxed{H(j\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt}$$

Frequency Response of LTI System given by $h(t)$.

Representation of the System

$h(t) \longleftrightarrow H(j\omega) \longrightarrow$

Time Domain — Frequency Domain

$H(j\omega) = |H(j\omega)|\, e^{j\measuredangle H(j\omega)}$: in polar form

ω: a continuous variable in $(-\infty, \infty)$.

Properties of $H(j\omega)$:

① CT Freq Response is Not periodic w/ 2π (Recall DT Freq.)

$$H(j(\omega+2\pi)) \neq H(j\omega), \quad \omega \in (-\infty, \infty).$$

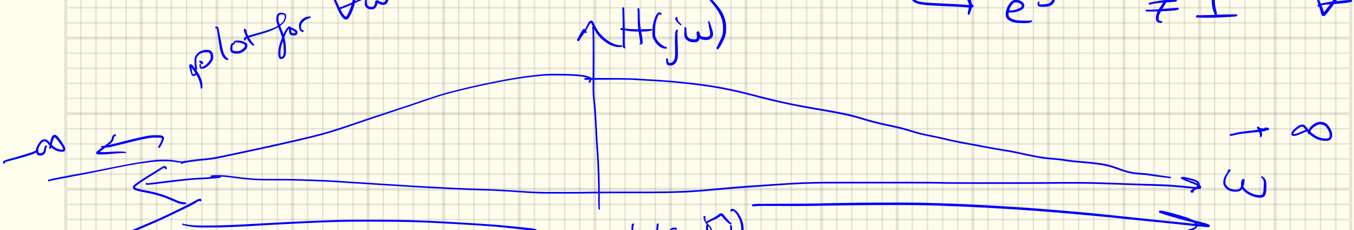
Apply the defn:

$$H(j(\omega+2\pi)) = \int_{-\infty}^{\infty} h(t)\, \underbrace{e^{-j(\omega+2\pi)t}}_{e^{-j\omega t}\, \cdot\, e^{-j2\pi t}}\, dt$$

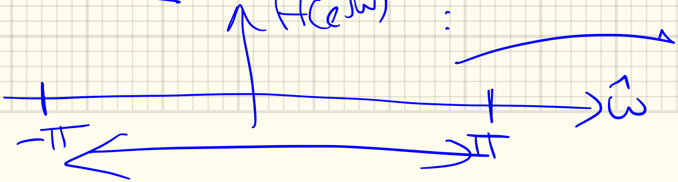
$\rightarrow e^{\mp j2\pi t} \neq 1 \quad \forall t.$

★ b/c now t is a continuous time variable

plot for $\forall\omega$



Recall in DT:



b/c $H(e^{j\hat{\omega}})$: periodic w/ 2π.

same as before

2) For a real sinusoid:

$$x(t) = A\cos(wt+\phi) \longrightarrow \boxed{H(jw)} \longrightarrow y(t) = A\cdot|H(jw)|\cdot\cos(wt+\phi+\angle H(jw))$$

③ If $h(t)$ is real $\rightarrow$ $H(-jw) = H^*(jw)$: we have Conjugate symmetry in the frequency resp.

$$\Rightarrow |H(jw)| = |H(-jw)|$$ even symmetry

$$\Rightarrow \angle H(jw) = -\angle H(-jw)$$: odd symmetry

4) Superposition rule: Sum of sinusoids inputs $\longrightarrow \boxed{H(jw)} \longrightarrow$ Sum of sinusoid outputs

$$x(t) = \sum_{k=1}^{N} A_k\cos(w_k t+\phi_k) \longrightarrow \boxed{H(jw)} \longrightarrow y(t)$$

$$y(t) = \sum_{k=1}^{N} A_k\,|H(j\underline{w_k})|\cdot\cos(w_k t+\phi_k+\angle H(jw_k))$$

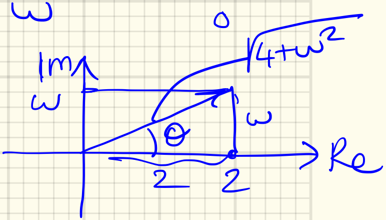
Ex: Given LTI system : $h(t) = 2e^{-2t}u(t)$ → real impulse response

$$H(jw) = ?$$

$$H(jw) = \int_{-\infty}^{\infty} h(t)\, e^{-jwt}\, dt = \int_{-\infty}^{\infty} 2e^{-2t}\underbrace{u(t)}\, e^{-jwt}\, dt$$

$$= \int_{0}^{\infty} 2\underbrace{e^{-(2+jw)t}}\, dt = 2(-)\frac{e^{-(2+jw)t}}{2+jw}\Bigg|_{0}^{\infty}$$

$$H(jw) = \frac{-2}{2+jw}(0-1) = \frac{2}{2+jw} \Rightarrow$$

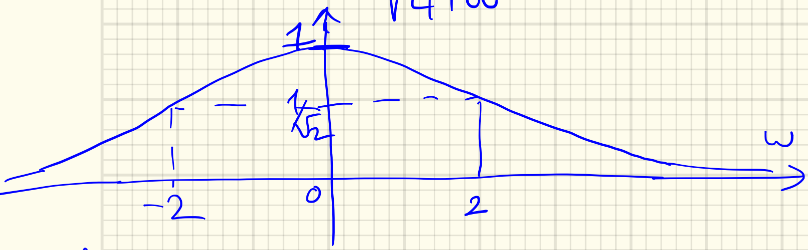


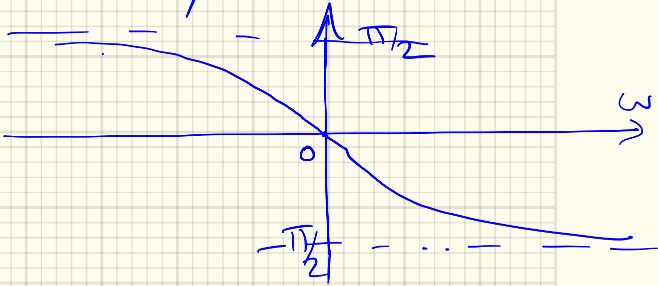
$$|H(jw)| = \frac{2}{\sqrt{4+w^2}}$$

$$\measuredangle H(jw) = \underbrace{\measuredangle 2}_{0} - \underbrace{\measuredangle(2+jw)}_{-\operatorname{atan}\left(\frac{w}{2}\right)} = -\operatorname{atan}\left(\frac{w}{2}\right)$$

$$|H(j\omega)| = \frac{2}{\sqrt{4+\omega^2}}$$

$$\measuredangle H(j\omega) = -\operatorname{atan}\left(\frac{\omega}{2}\right)$$





even symmetric

Note: you may see log-log plot

Q: What type of a filter?

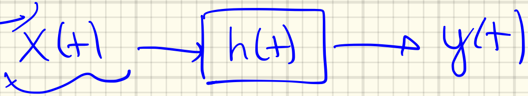
Look at the magnitude response shape

Low pass Filter

Ex: Use superposition principle to find the output of the LTI system

$$\underbrace{h(t) = e^{-2t}u(t)}$$

$$H(j\omega) = \frac{1}{2+j\omega}$$



Given input

Let $x(t) = \underbrace{5}_{x_1(t)} + \underbrace{8\cos\left(2t+\frac{\pi}{3}\right)}_{x_2(t)} + \underbrace{3\,\delta(t-0.1)}_{x_3(t)}$

$x_1(t)$: $\omega_1 = 0 \rightarrow y_1(t) = 5.\underbrace{H(j.0)}_{\frac{1}{2}} = \frac{5}{2}$

$x_2(t)$: $\omega_2 = 2\,\frac{rad}{s} \rightarrow H(j\underset{?}{2}) = \frac{1}{2+j2} \rightarrow |H(j2)| = \frac{1}{\sqrt{4+4}} = \frac{1}{2\sqrt{2}}$

$\measuredangle H(j2 = 0 - \text{atan}\left(\frac{2}{2}\right)$

$= -\pi/4$

$\rightarrow y_2(t) = 8.\frac{1}{2\sqrt{2}}\cos\left(2t+\frac{\pi}{3}-\frac{\pi}{4}\right)$

$x_3(t) = 3\,\underset{?}{\delta}(t-0,1) \rightarrow y_3(t) = x_3(t) * h(t)$

easier to do convolution (∴ time domain processing is preferred for the 3rd signal component)

$y_3(t) = 3\,\delta(t-0,1) * h(t) = 3\,h(t-0,1)$

$y_3(t) = 3\,e^{-2(t-0,1)}\,u(t-0.1)$

$\rightarrow y(t) = y_1(t) + y_2(t) + y_3(t)$

Ex: $x(t) = \underbrace{2\cos t}_{x_1(t)} + \underbrace{\cos(3t + 1.57)}_{x_2(t)}$

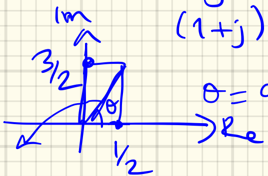
Given $H(j\omega) = \dfrac{2+j\omega}{1-j\omega}$: $x(t) \rightarrow \boxed{H(j\omega)} \rightarrow y(t) = ?$

Use superposition to find the output $y(t)$

$x_1(t) = 2\cos t$: $\omega_1 = 1$ rad/s $\rightarrow H(j1) = \dfrac{2+j}{1-j}$ $(1+j)$

$$H(j1) = \frac{2+j2+j+j^2}{1-j^2} = \frac{1+j3}{2}$$

polar form $\simeq \boxed{1.58e^{-j0.32}}$



$\theta = \text{atan}\left(\frac{3/2}{1/2}\right)$

$x_2(t)$: $\omega_2 = 3$ rad/s $\quad H(j3) = \dfrac{2+j3}{1-j3} \simeq \boxed{1.14\,e^{-j0.25}}$

(use a calculator)

exercise Check this

$\rightarrow y_1(t) = 2\underbrace{|H(j1)|}_{1.58}\cos(t - 0.32) = 2(1.58)\cos(t - 0.32)$

$y_2(t) = \underbrace{|H(j3)|}_{1.14}\cos(3t + 1.57 - 0.25) = 1.14\cos(3t + 1.32)$

$\Rightarrow y(t) = y_1(t) + y_2(t)$ due superposition.

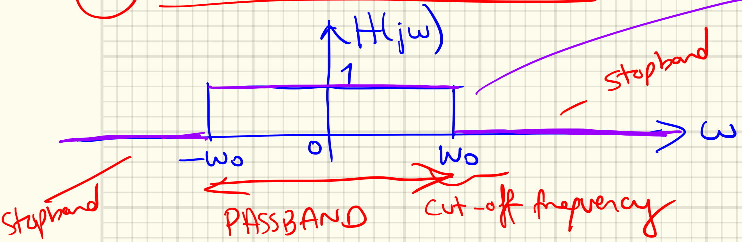
IDEAL FILTERS ; Frequency-Selective Systems

(i) have value of 1 in the passband (desired frequencies)

(ii) have value of 0 in the stopband

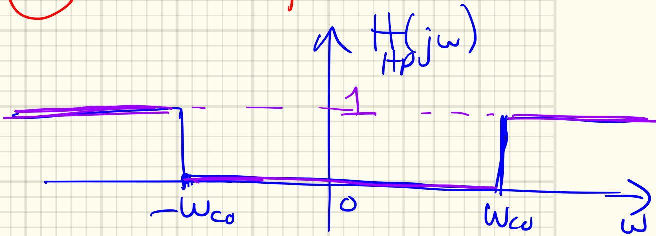
No transition band in an ideal filter.

① Ideal LowPass Filter:



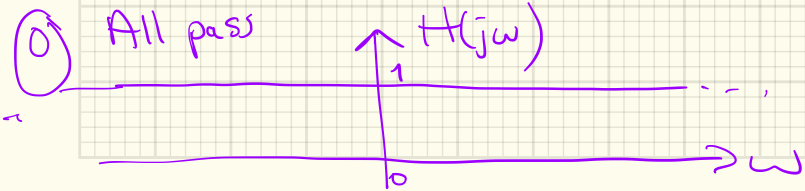
$$H_{LP}(j\omega) = \begin{cases} 1 & , |\omega| < \omega_{co} \\ 0 & , |\omega| > \omega_{co} \end{cases}$$

② Ideal HighPass Filter:



$$H_{HP}(j\omega) = 1 - H_{LP}(j\omega)$$

⓪ All pass



③ Ideal Delay System:

$y(t) = x(t-t_0)$

$h(t) = \delta(t-t_0)$

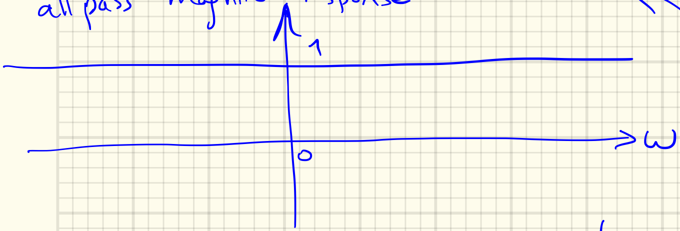
$$H(j\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$$

$$= \underbrace{\int_{-\infty}^{\infty} \underbrace{\delta(t-t_0)}\, \underbrace{e^{-j\omega t}}\, dt}$$

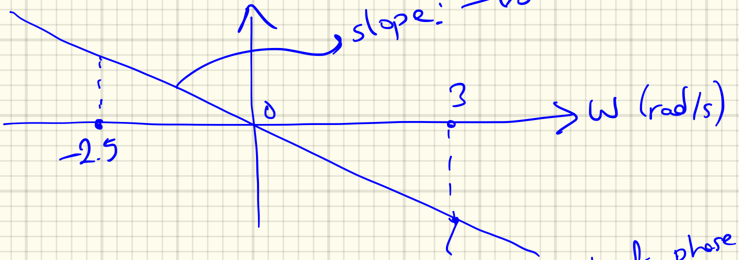
$$\boxed{H(j\omega) = e^{-j\omega \overset{\uparrow}{t_0}}}$$

$$|H(j\omega)| = 1$$

all pass magnitude response



Phase Response $\measuredangle H(j\omega) = -\omega t_0$

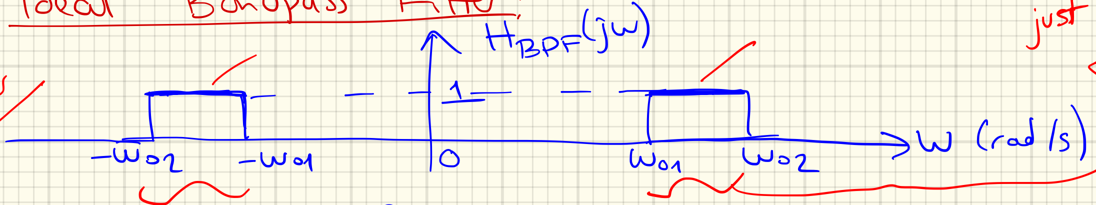


$-3t_0$: amount of phase shift it introduces

Recall this is a Linear Phase System

④ Ideal Bandpass Filter:

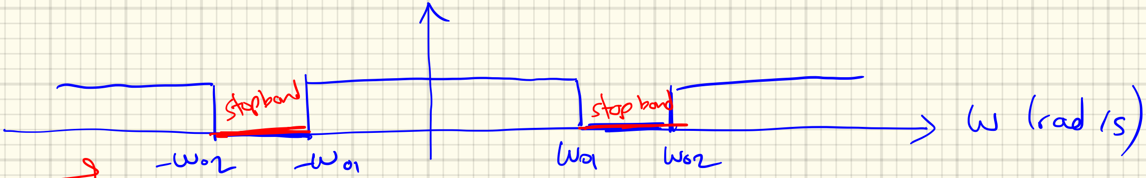
Recall: Audio equalizer



just passes these frequencies, eliminates everything else.

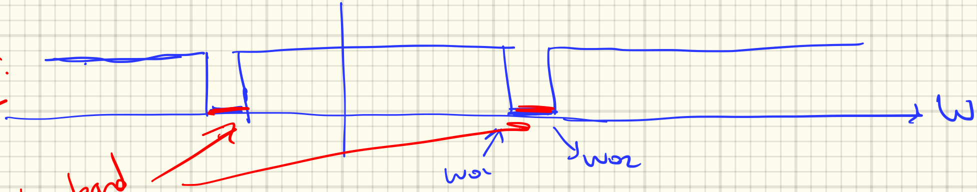
$$H_{BPF}(j\omega) = \begin{cases} 1 & , \omega_{01} < |\omega| < \omega_{02} \\ 0 & , \text{elsewhere} \end{cases}$$

⑤ Ideal Bandstop Filter: $H_{BSF}(j\omega) = 1 - H_{BPF}(j\omega)$



Cutoff Frequencies: parameters to be selected

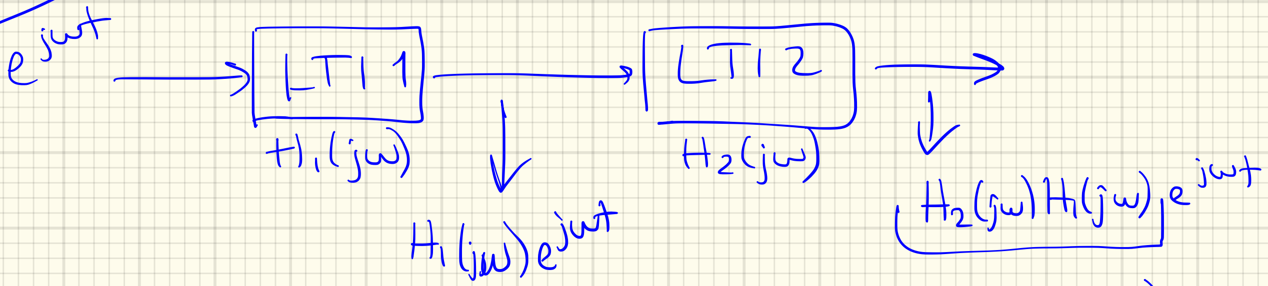
Notch filter:



Very narrow stop band

Cascade & Parallel Connection of LTI Systems (Rules are same before)

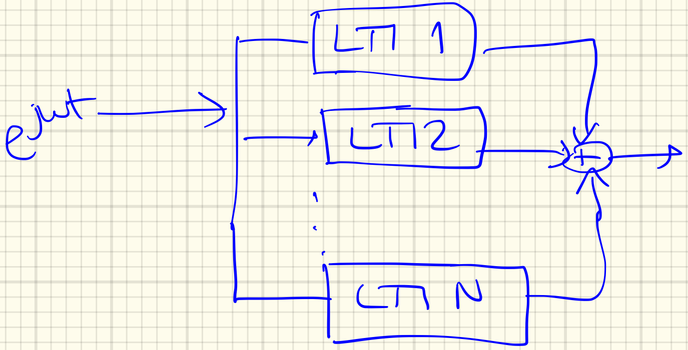
Cascade



→ Overall response : $H_0(j\omega) = H_1(j\omega)\ H_2(j\omega) \dots H_N(j\omega)$

for N filters in cascade

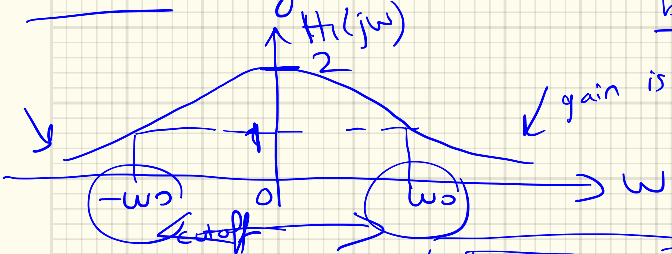
Parallel



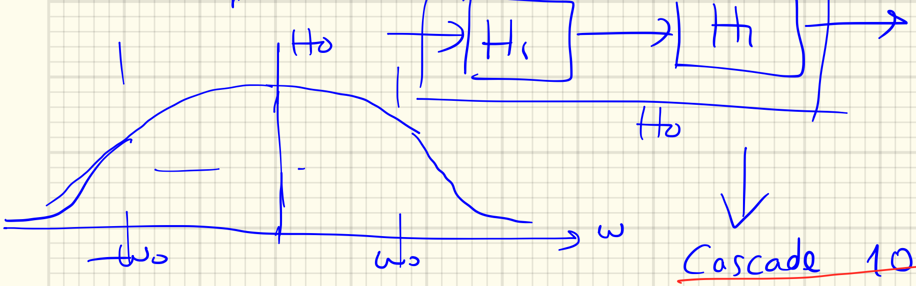
$$\underbrace{\left(H_1(j\omega) + H_2(j\omega) + \dots + H_N(j\omega)\right)}_{H_0(j\omega)} e^{j\omega t}$$

Ex: Say we have a "cheap" LP filter w/ H(jω)

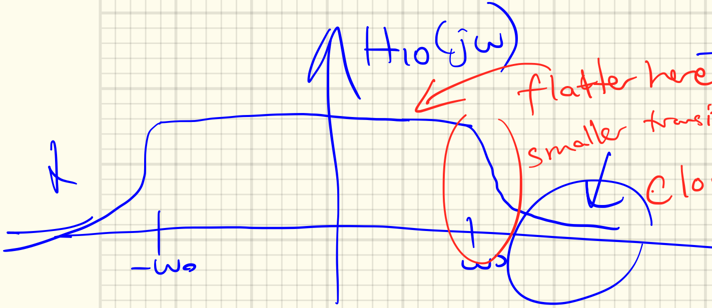
bad



gain is not as small as I want



Cascade 10 of these filter



flatter here

smaller transition

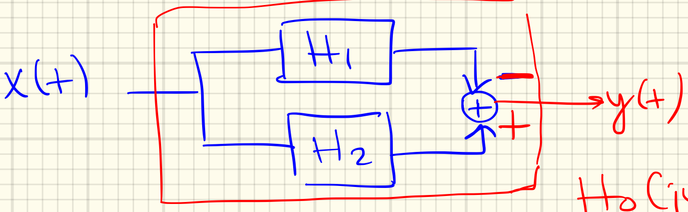
Close to zero in stop band

ω

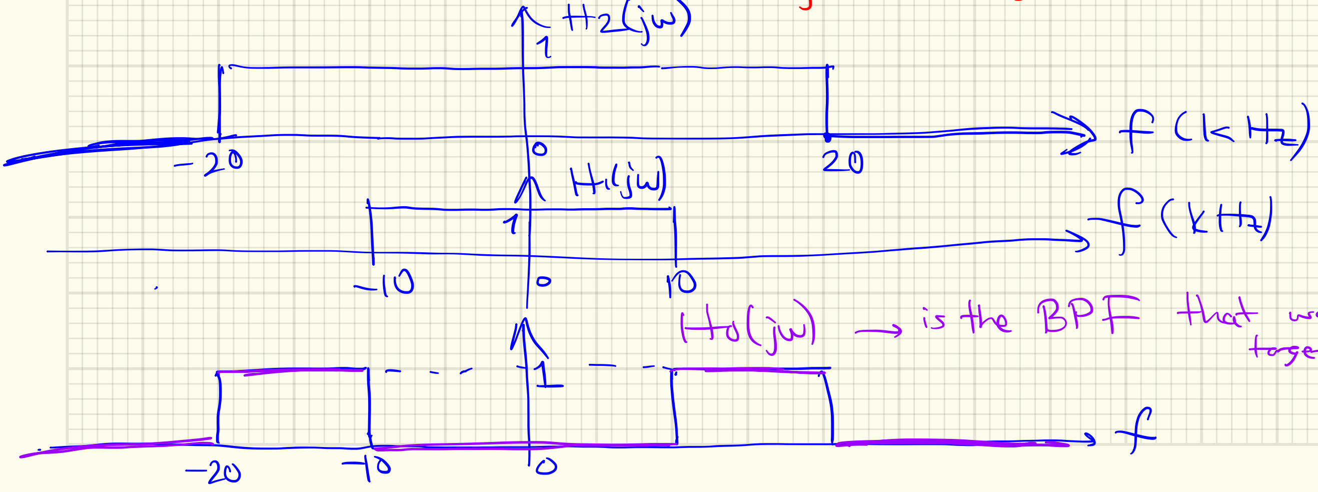
$H_{10}(j\omega)$: I get a "better" LP filter.

Ex: You have 2 cheap LP filters w/ cutoff frequencies at $f_1 = 10\,\text{kHz}$
$f_2 = 20\,\text{kHz}$.

Q: Make a band pass filter w/ passband in (10, 20) kHz



$$H_0(j\omega) = H_2(j\omega) - H_1(j\omega)$$



$H_0(j\omega)$ → is the BPF that was targeted.