BLG 354E Signals & Systems

CT Fourier Transform

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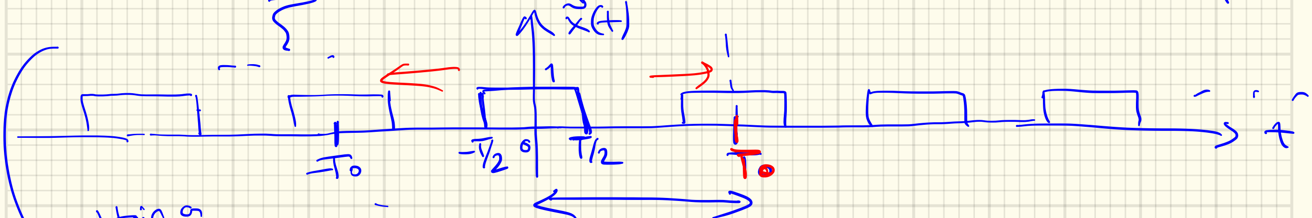
CHAPTER 11 Continuous Time (CT) Fourier Transform (SPFirst book)

Goal: Want to develop a general definition of frequency spectrum for any signal $x(t)$ (both non-periodic & periodic)

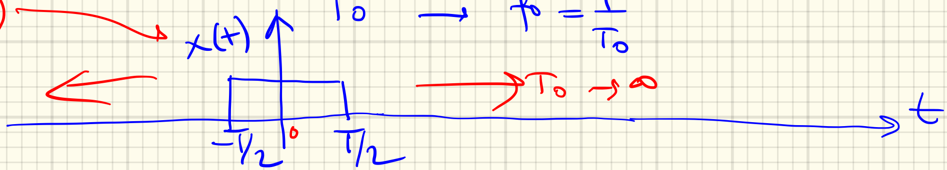
Fourier series representation

Recall for a periodic signal, take $\tilde{x}(t)$: any periodic signal.



$T_o \rightarrow f_o = \frac{1}{T_o}$

obtain a non-periodic signal



$$\boxed{x(t) = \lim_{T_o \to \infty} \tilde{x}(t)}$$

F.S. expansion of $\tilde{x}(t)$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t} \rightarrow \omega_o = 2\pi f_o$$

F.S. coeff. for $\tilde{x}(t)$:

$$a_k = \frac{1}{T_o}\int_{-T_o/2}^{T_o/2} \tilde{x}(t) \cdot e^{-jk\omega_o t}\, dt = \frac{1}{T_o}\int_{-T/2}^{T/2} 1 \cdot e^{-jk\omega_o t}\, dt = \frac{\sin(\omega_o k T/2)}{T_o\, k\, \frac{\omega_o}{2}} \Rightarrow$$

$$\underbrace{a_k \cdot T_0}_{\text{for periodic } \tilde{x}(t)} = \frac{\sin(k\,\omega_0 T/2)}{k\,\omega_0/2}$$

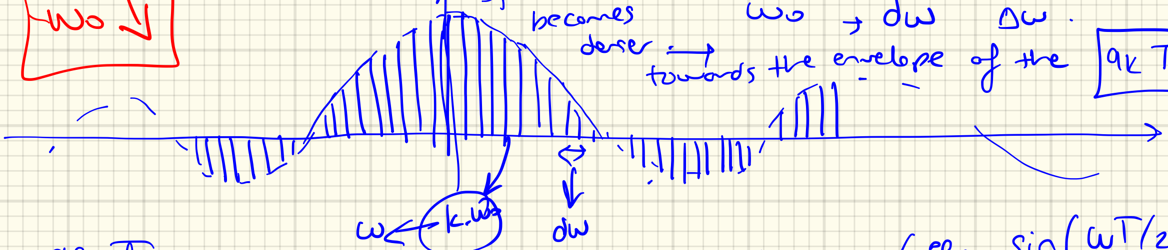
$\boxed{a_k \cdot T_0}$

$T_0 \uparrow$ $\boxed{\omega_0 \downarrow}$

spectrum becomes denser $\longrightarrow$ towards the envelope of the $\boxed{a_k T_0}$

$T_0 \to \infty$

$\omega_0 \to d\omega$ $\Delta\omega$.



as $T_0 \to \infty$, $a_k T_0 \to$ envelope $\left(\text{eg. } \dfrac{\sin(\omega T/2)}{\omega/2}. \right) \equiv$ CT. Fourier transform of $x(t)$

$\omega_0 \to d\omega$

$k.\omega_0 \to \omega$

$\hookrightarrow$ cont. frequency variable

$\Longrightarrow$

$$\underbrace{a_k \cdot T_0}_{\text{envelope}} = \int_{-T_0/2}^{T_0/2} \tilde{x}(t)\, e^{-jk\omega_0 t}\, dt \qquad \lim_{T_0 \to \infty} \tilde{x}(t) = \underset{\text{nonperiodic}}{x(t)}$$

$T_0 \to \infty$

CT. Fourier Transform of $\underset{\text{non-periodic}}{x(t)}$

$$\boxed{X(j\omega) \triangleq \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt}$$

CT. Fourier Transform. (CTFT) of $x(t)$

Def: CTFT of $\underline{x(t)}$: $X(j\omega) \triangleq \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t}\, dt$

☆ $x(t)$ is not a periodic signal.

It has a F.T. not a F. Series (Fourier)

☆ $\underline{X(j\omega)}$: shows the "frequency content" of $x(t)$.

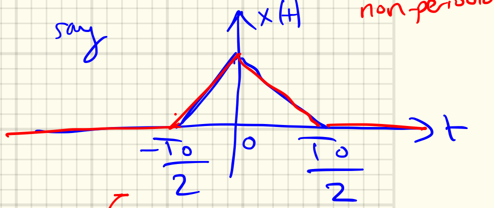
Fourier Transform is a Spectrum representation of $x(t)$.

Time Domain: $x(t)$ $\xrightarrow{\text{Fourier Xfrm}}$ Frequency Domain: $X(j\omega)$

$\xleftarrow{\text{Inverse F.T.?}}$

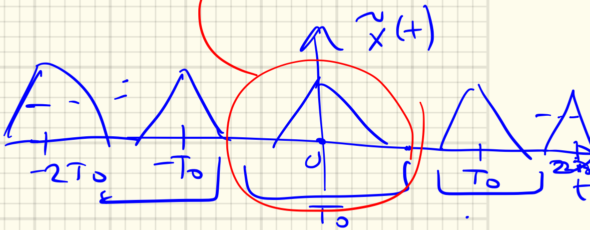
How to go back from $X(j\omega)$ to $x(t)$?

Let's say $X(j\omega)$ is the F.T. of $x(t)$. say



Construct a periodic extension of $x(t)$:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x(t - k.T_0)$$



Find F.S. coeff

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \tilde{x}(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T_0} \underbrace{\int_{\cancel{-T_0/2}\ -\infty}^{\cancel{T_0/2}\ \infty} x(t)\, e^{-jk\omega_0 t}\, dt}_{X(jk\omega_0)} = \frac{1}{T_0} X(jk\omega_0)$$

$$\rightarrow \boxed{a_k . T_0 = X(jk\omega_0)}$$

Relation btw F.S. coef of a periodic signal & the F.T. of ~~the~~ corresp. signal in 1 period & zero everywhere else

Q. Can we obtain $x(t)$ from $X(j\omega)$? i.e. Inverse F.T.

$$\rightarrow \tilde{x}(t) = \boxed{\sum_{k=-\infty}^{\infty} \underbrace{a_k}\, \underbrace{e^{jk\omega_0 t}}}$$ F.S. expansion.

$$= \sum_{k} \frac{X(jk\omega_0)}{T_0} e^{jk\omega_0 t}$$

insert here what we've just calculated.

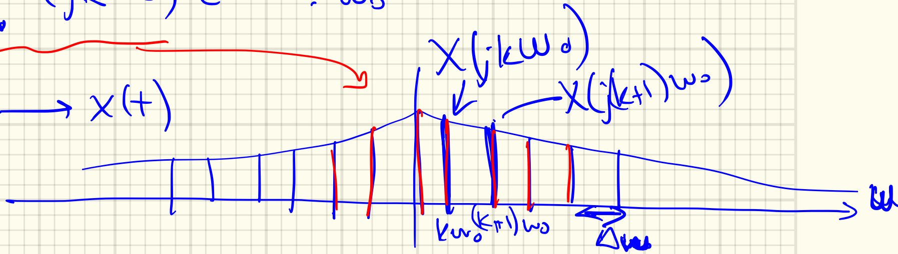
$$T_0 = \frac{2\pi}{\omega_0}$$

$$\tilde{x}(t) = \frac{1}{2\pi} \underbrace{\sum_{k=-\infty}^{\infty} X(jk\omega_0)\, e^{jk\omega_0 t} \cdot \omega_0}$$

as $T_0 \to \infty$: $\tilde{x}(t) \longrightarrow x(t)$

$\omega_0 \to d\omega \to 0$

$k\omega_0 \to \omega$



$$\boxed{\underbrace{x(t)} \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{X(j\omega)}_{\sim \text{spectrum coeff.}}\, \underbrace{e^{j\omega t}}_{\text{basis fn}}\, d\omega}$$

Inverse (CT) Fourier Transform

For a non-periodic signal

$X(j\omega) \longrightarrow x(t)$. $\Longrightarrow$

Fourier Transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x(t) \xrightarrow{\text{F.T.}} X(j\omega)$$

$$x(t) \xleftarrow{\text{I.F.T.}} X(j\omega)$$

1-1 relationship.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

Time ... Frequency.

Existence of F.T. Sufficient condition : F.T. exists when

$$\underbrace{\int_{-\infty}^{\infty} |x(t)|\, dt}_{\text{this integral is finite.}} < \infty .$$

(not a necessary condn).

$\equiv$

If the signal is absolutely integrable its F.T. exists.

but $\exists$ signals that are not absolutely integral ($x(t) \neq \cos(\omega t)$). but its F.T. exists.

$x(t) = u(t)$.

→ We start to develop a library of F.T. pairs :

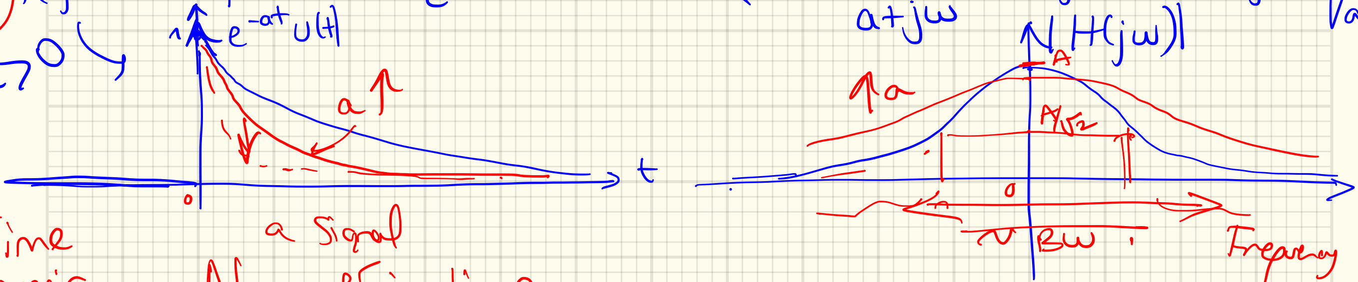
Ex : $x(t) = e^{-5t}u(t)$ → its F.T. ?

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-5t}u(t)\, e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-(5+j\omega)t}\, dt = \frac{1}{5+j\omega}$$

$$x(t) = e^{-5t}u(t) \Longleftrightarrow \frac{1}{5+j\omega} = X(j\omega)$$

① Right-sided exponential $\quad e^{-at}u(t) \Longleftrightarrow \frac{1}{a+j\omega} = H(j\omega) \rightarrow |H(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$

$a > 0$



Narrower in time → Wider spectrum. (BW is larger)

Wider in time ← Narrower in frequency

Duality btw. Time & freq. domain

→ due to Heisenberg Uncertainty Principle: →

Signal is Wide in Time Domain $\Longleftrightarrow$ Signal is Narrow in Frequency domain (BW $\downarrow$)

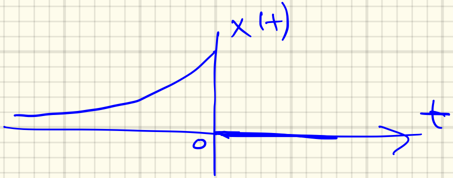
" Narrow " $\Longleftrightarrow$ Wide "

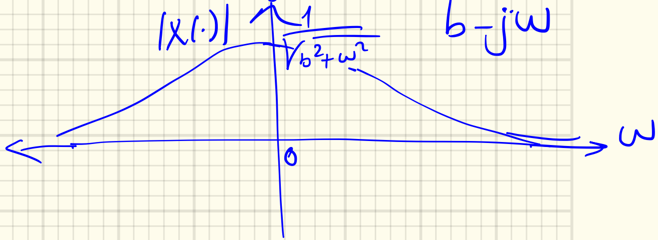
Signal (BW $\uparrow$)

Bandwidth $\uparrow$

Note: exercise Show for a left-sided exponential signal, its F.T is

$$x(t) = e^{bt}\, u(-t) \quad , b>0 \longrightarrow X(j\omega) = \frac{1}{b-j\omega}$$





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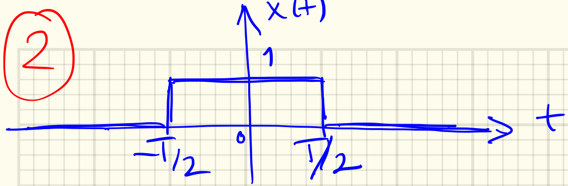
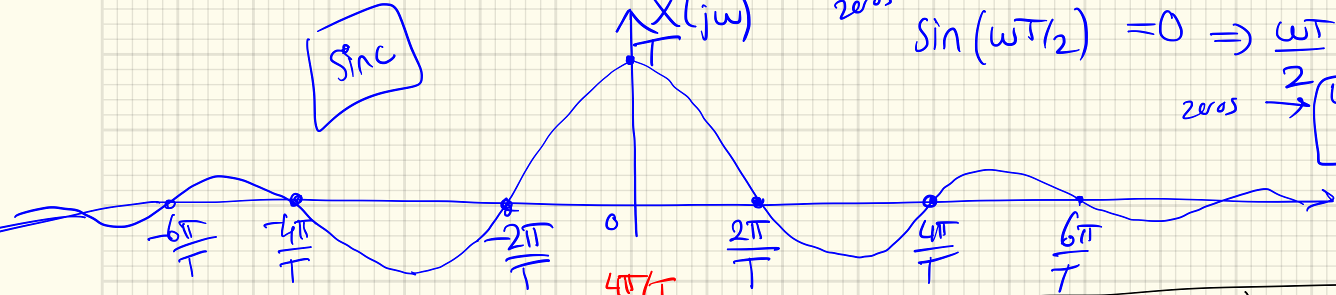


$$\longleftrightarrow \quad X(j\omega) = ?$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-T/2}^{T/2} 1 \cdot e^{-j\omega t}\, dt = \frac{e^{-j\omega t}}{-j\omega}\Bigg|_{-T/2}^{T/2}$$

$$\dashrightarrow X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2} \quad \rightarrow \omega = 0 \quad \frac{\frac{T}{2}\cos(\omega T/2)}{\cancel{1}/2} = T$$

use l'hospital



zeros

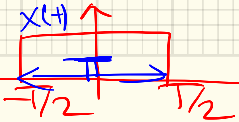
$$\sin(\omega T/2) = 0 \Rightarrow \frac{\omega T}{2} = \pi k$$

zeros $\rightarrow \boxed{\omega = \frac{2\pi k}{T}}$

$4\pi/T$

~ BW : width of mainlobe

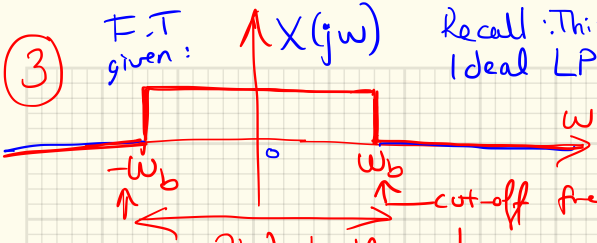
Note: $x(t)$: $T \uparrow$



$T \uparrow$ (wider in time) $\longleftrightarrow$ narrower in freq $BW \downarrow$

$T \downarrow \longleftrightarrow BW \uparrow$ (freq)

(3) F.T given:



Recall: This is an Ideal LPF (Low Pass filter).

eg. → $H(j\omega)$: frequency response

→ $x(t) = ?$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\omega_b}^{\omega_b} 1 \cdot e^{j\omega t}\, d\omega = \frac{1}{2\pi}\, \frac{e^{j\omega t}}{jt}\Bigg|_{-\omega_b}^{\omega_b}$$

$$= \frac{1}{\pi t}\left(\frac{e^{j\omega_b t} - e^{-j\omega_b t}}{2j}\right) \Rightarrow x(t) = \frac{\sin(\omega_b t)}{\pi t}$$

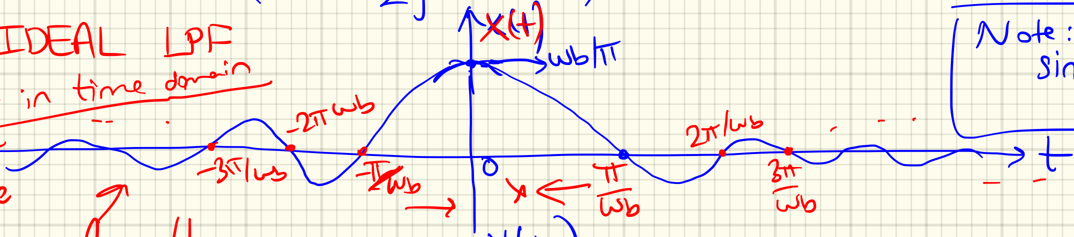
Note: $\text{sinc}(x) \triangleq \frac{\sin(\pi x)}{\pi x}$

For the IDEAL LPF

→ filter in time domain

≡ impulse response

is ∞ - length ∴

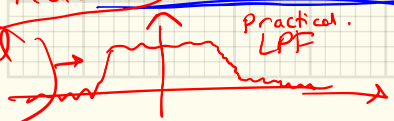


∴ Ideal filter is not realizable.

BW ↑ (ω_b ↑) wider in freq. domain ⟷ narrower in time domain

narrow in freq. domain ⟷ wide in Time

Practical filter

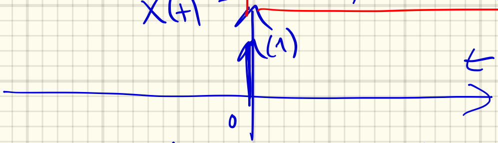


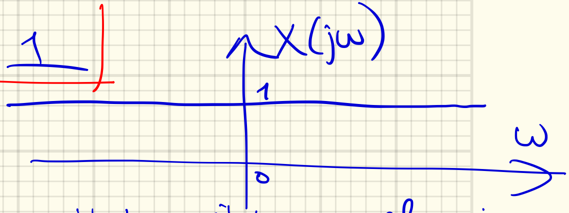
4. F.T. $\{\delta(t)\} = ?$

$$\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \delta(t).\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \delta(t).1.\, dt = 1.$$

samples at $t=0$

$$\boxed{\delta(t) \xleftrightarrow{\text{F.T}} 1}$$

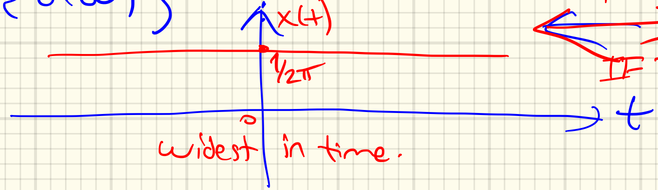




narrowest possible signal in time

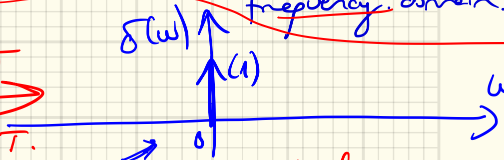
widest possible waveform in frequency domain.

5. $\text{F.T}^{-1}\{\delta(\omega)\} = ?$



F.T

I.F.T.



widest in time.

narrowest in freq.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int \delta(\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)\, d\omega = \frac{1}{2\pi}$$

this signal has freq. $=0$

$$\boxed{1 \longleftrightarrow 2\pi.\delta(\omega)}$$

Time Freq.

6 $$\boxed{\delta(t-t_0) \xleftrightarrow{F.T.} e^{-j\omega t_0}}$$

7 $$\frac{1}{2\pi} e^{j\omega t_0} \xleftrightarrow{F.T} \delta(\omega-\omega_0)$$

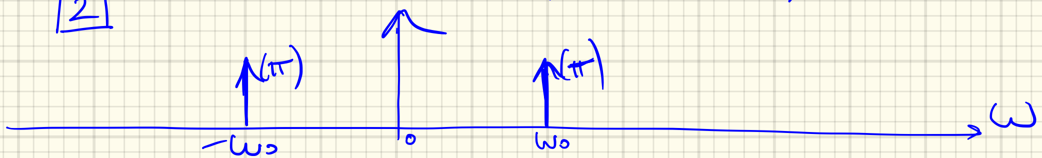
show: $$F.T.^{-1}\{\delta(\omega-\omega_0)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega-\omega_0)\, e^{j\omega t}\, d\omega$$

$$\frac{1}{2\pi}\int \delta(\omega-\omega_0)\, e^{j\omega_0 t}\, d\omega = \frac{e^{j\omega_0 t}}{2\pi}$$

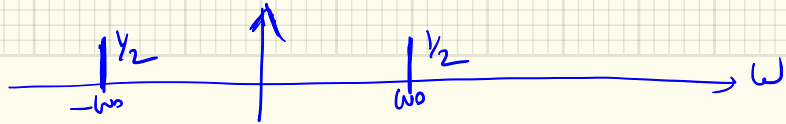
$$\boxed{e^{j\omega_0 t} \xleftrightarrow[I.F.T]{F.T} 2\pi \cdot \delta(\omega-\omega_0)}$$

8 $$x(t) = \cos(\omega_0 t) \longrightarrow F.T.\ ?$$

$$x(t) = \boxed{\frac{1}{2}} e^{j\omega_0 t} + \boxed{\frac{1}{2}} e^{-j\omega_0 t} \longrightarrow \pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0)$$



Recall freq. (Fourier series) spectrum of $\cos(\omega_0 t)$ $\longrightarrow$

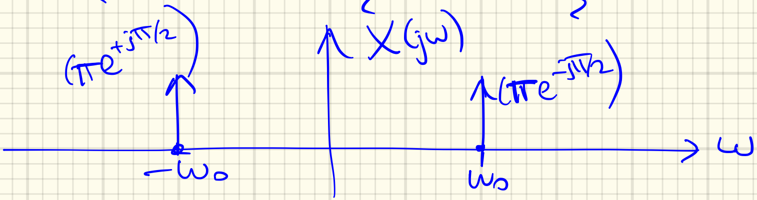


exercise:

$$x(t) = \sin(\omega_0 t) \longrightarrow F.T\{x(t)\}$$

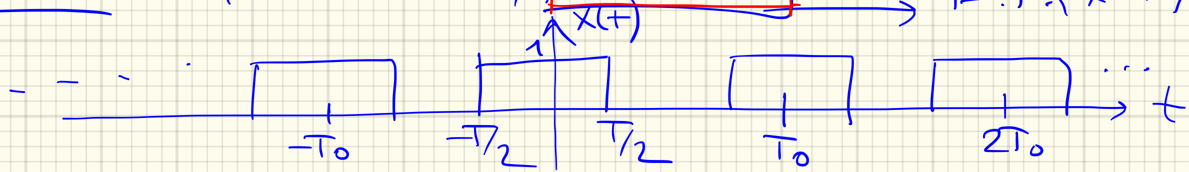
$$= \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right) \longrightarrow \pi e^{-j\pi/2}\,\delta(\omega-\omega_0) + \pi e^{j\pi} e^{-j\pi/2}\,\delta(\omega+\omega_0)$$

Recall $e^{j\omega_0 t} \longleftrightarrow 2\pi\,\delta(\omega-\omega_0)$



Note: For periodic signals, $X(j\omega)$ the F.T. consists of weighted impulses at harmonic frequencies of ω_0, $(k\omega_0)$ where their weights $= 2\pi \cdot a_k$.

Ex: Square Wave / $\boxed{\text{Periodic}}$ $\longrightarrow$ $F.T.\{x(t)\} = ?$



$$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{2T} = \pi/T$$

$T_0 = 2T$

Recall we did this before: F.S. coef for x(t).

$$a_k = \begin{cases} \dfrac{\sin(\pi k/2)}{\pi k}, & k \neq 0, \ k \text{ odd} \\ 1/2, & k = 0 \\ 0, & k \text{ even} \end{cases}$$

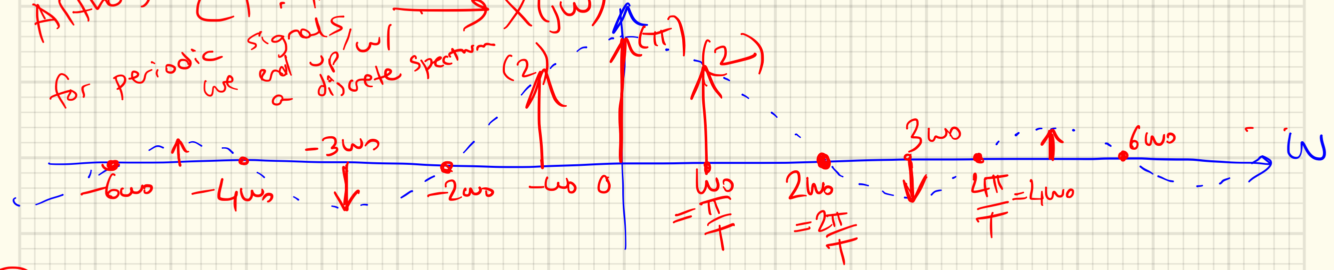
F.T of x(t) periodic $\longrightarrow$

$$\boxed{X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \, \delta(\omega - k\omega_0)}$$

fund. freq.

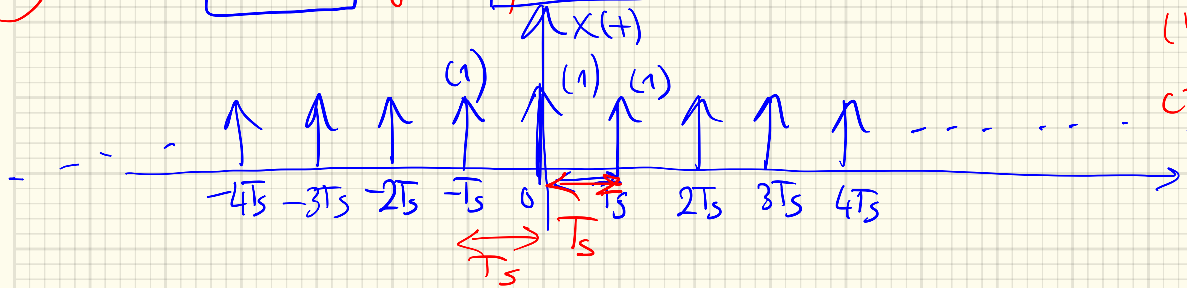
$$\Rightarrow \quad X(j\omega) = \pi\,\delta(\omega) + \sum_{\substack{k=-\infty \\ k\neq 0,\ k\ \text{odd}}}^{\infty} 2\cancel{\pi} \cdot \underbrace{\frac{\sin(\pi k/2)}{\cancel{\pi} k}}_{a_k}\,\delta(\omega - k\omega_0)$$

Although this is CT F.T. for periodic signals, we end up w/ a discrete spectrum



(10) Find $\boxed{\text{F.T}}$ of $\boxed{\text{periodic}}$ impulse train signal ?

(used for sampling CT signals)



$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$

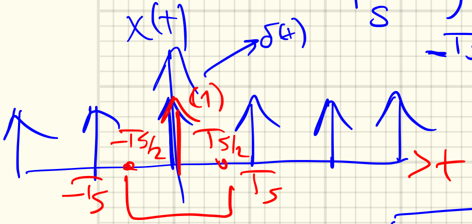
is periodic w/ $T_0 = T_s$ $\rightarrow$

→ b/c $x(t)$ is periodic signal, we first calculate a_k (F.S. coeff) in order to find $F.T\{x(t)\}$.

$$a_k = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \delta(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T_s}\underbrace{\int_{-T_s/2}^{T_s/2} \delta(t)\cdot 1\, dt}_{1} = \frac{1}{T_s}.$$

1 :(used sampling prop of $\delta(t)$)



$$\Rightarrow \boxed{a_k = \frac{1}{T_s}}$$

$T_s \uparrow$: wider in time, narrower in freq

& vice versa.

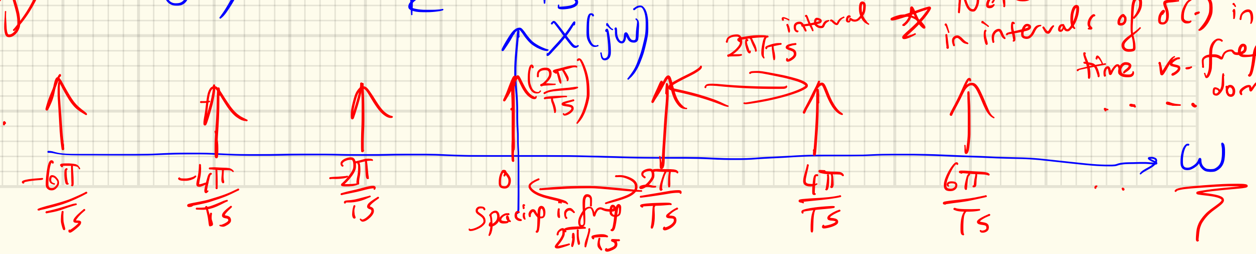
$$\Rightarrow X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k\, \delta(\omega - k\omega_0)$$

$$\omega_0 = \frac{2\pi}{T_s}$$

$$X(j\omega) = \sum_k \frac{2\pi}{T_s}\, \delta\left(\omega - k\frac{2\pi}{T_s}\right)$$

another impulse train in Frequency Domain

☆ Note the inverse relation in intervals of $\delta(\cdot)$ in time vs. freq domain



Properties of CT F.T.

(Note Table 11.2 lists some F.T. pairs)

1) Linearity:

$$x(t) \longleftrightarrow X(j\omega)$$
$$y(t) \longleftrightarrow Y(j\omega)$$

$$a x(t) + b\, y(t) \longleftrightarrow a X(j\omega) + b\, Y(j\omega)$$

show

2) Time-Shifting:

$$x(t) \longleftrightarrow X(j\omega)$$
$$x(t-t_0) \longleftrightarrow e^{-j\omega t_0} X(j\omega)$$

proof: $$\int_{-\infty}^{\infty} x(\underbrace{t-t_0}) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(\tau) e^{-j\omega \tau} \boxed{e^{-j\omega t_0}} d\tau = e^{-j\omega t_0} X(j\omega)$$

$\tau = t - t_0$

$t = \tau + t_0$

3) Frequency-Shifting:

$$x(t) \longleftrightarrow X(j\omega)$$
$$e^{j\omega_0 t} x(t) \longleftrightarrow X(j(\omega-\omega_0))$$

pf: $$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j(\underbrace{\omega-\omega_0}))e^{j\omega t} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\sigma)\, e^{j(\sigma+\omega_0)t} d\sigma \longrightarrow$$

$e^{j\sigma t} / \boxed{e^{j\omega_0 t}}$

$\sigma = \omega - \omega_0 \rightarrow \omega = \sigma + \omega_0$

$d\sigma = d\omega$

$$\Rightarrow \; = \frac{1}{2\pi} e^{j\omega_0 t} \underbrace{\int_{-\infty}^{\infty} X(j\sigma)\, e^{j\sigma t}\, d\sigma}_{x(t)} = e^{j\omega_0 t} \cdot x(t).$$

4 Time Scaling:

$$x(t) \longleftrightarrow X(j\omega)$$

$$x(at) \longleftrightarrow \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$$

pf:

$$\int_{-\infty}^{\infty} x(\underbrace{at}) e^{-j\omega t} dt = \frac{1}{|a|} \underbrace{\int_{-\infty}^{\infty} x(\tau) e^{-j\left(\frac{\omega}{a}\right)\tau} d\tau}$$

$\tau = at$

$d\tau = |a| dt$

$$= \frac{1}{|a|} X\left(j\left(\frac{\omega}{a}\right)\right)$$

Recall relation btw Time & Freq.

$$x(2t) \longleftrightarrow \underbrace{\frac{1}{2} X\left(j\frac{\omega}{2}\right)}$$

narrower in time (compressed) $\longleftrightarrow$ wider spectrum in freq.

$$x\left(\frac{t}{3}\right) \longleftrightarrow 3\, X(j\,3\omega)$$: narrower in frequency

wider in time

5 Time Flip : $x(-t) \longleftrightarrow X(-j\omega)$

We use previous property: $x(at) \longleftrightarrow \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$

w/ $\boxed{a=-1}$

6 In Freq: $\underbrace{X(j0)}_{\text{F.T at } \omega=0} = \int_{-\infty}^{\infty} x(t)\, \underbrace{e^{-j\omega t}}_{=1 \,\leftarrow\, \omega=0}\, dt \Big| = \underbrace{\int_{-\infty}^{\infty} x(t)\, dt}_{\text{Area under the signal } x(t).}$

In time: $x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, \underbrace{\cancel{e^{j\omega t}}}_{1}\, d\omega \Big|_{t=0} = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, d\omega}_{\text{Area under } \frac{X(j\omega)}{2\pi}}$