BLG 354E Signals & Systems

24.05.2021

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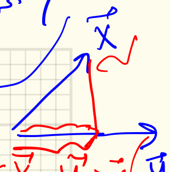
Last time

We defined Fourier transform of a signal:

Given $x(t)$:

$$\text{F.T of } x(t):\ X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \Rightarrow \langle x(t), e^{-j\omega t} \rangle \quad (F)\ \text{F.T}$$



$\langle \vec{x}, \vec{y} \rangle \vec{y}$

Given $X(j\omega)$:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$\langle X(j\omega), e^{j\omega t} \rangle$$

We derived F.T. pairs ✓

You can always resort to look-up tables eg. 11.2, 11.3 in SP First book.

We were deriving F.T. properties

1) Linearity ✓
2) Time-shifting

$$x(t) \longleftrightarrow X(j\omega)$$

$$x(t-t_0) \longleftrightarrow e^{-j\omega t_0} X(j\omega)$$

3) Freq shifting

$$x(t) \longleftrightarrow X(j\omega)$$

$$e^{j\omega_0 t}\, x(t) \longleftrightarrow X(j(\omega-\omega_0))$$

4) Time Scaling

$$x(at) \longleftrightarrow \frac{1}{|a|} X\left(j\left(\frac{\omega}{a}\right)\right)$$

⋮

7) Convolution Property:

$$x(t) \longleftrightarrow X(j\omega)$$
$$y(t) \longleftrightarrow Y(j\omega)$$

$$\boxed{z(t) = x(t) * y(t)} \longleftrightarrow Z(j\omega) = ?$$

Show

$$Z(j\omega) = \int_{-\infty}^{\infty} z(t)\, e^{-j\omega t}\, dt = \underbrace{\int_{-\infty}^{\infty}}_{(t)\text{ integral}} \underbrace{\int_{-\infty}^{\infty}}_{(\tau)\text{ integral}} \underbrace{x(\tau)\, y(t-\tau)\, d\tau}_{\text{convolution}}\; e^{-j\omega t}\, dt$$

$$= \int_{(\tau)} x(\tau) \underbrace{\int_{(t)} y(t-\tau)\, e^{-j\omega t}\, dt}_{e^{-j\omega\tau} Y(j\omega) \text{ (from time shifting property)}}\, d\tau = \underbrace{\left(\int_{(\tau)} x(\tau)\, e^{-j\omega\tau}\, d\tau\right)}_{X(j\omega)} Y(j\omega)$$

$$\boxed{Z(j\omega) = X(j\omega)\cdot Y(j\omega)}$$

Convolution in Time Domain $\Longleftrightarrow$ pointwise Multiplication in Frequency Domain

8) Multiplication Property:

$$z(t) = x(t) \cdot y(t) \longleftrightarrow Z(j\omega) = \frac{1}{2\pi} X(j\omega) * Y(j\omega)$$

multiplication in time domain $\longleftrightarrow$ Convolution in Fourier Domain

→ exercise : show this.

9) Differentiation Property:

$$x(t) \longleftrightarrow X(j\omega)$$

$$\frac{d}{dt} x(t) \longleftrightarrow j\omega \cdot X(j\omega)$$

pf →

$$\frac{d}{dt}\left(x(t)\right) = \frac{d}{dt}\left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega\right)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \boxed{X(j\omega) \cdot j\omega} \cdot \underbrace{e^{j\omega t}}_{\frac{d}{dt} e^{j\omega t}}\, d\omega$$

$$= F.T.\{ j\omega X(j\omega) \}$$

Generalizes to:

$$\boxed{\frac{d^n}{dt^n} x(t) \longleftrightarrow (j\omega)^n X(j\omega)}$$

→ show this

→

→ We may be given I/O relationship in an LTI system; i.t.o. differential equations

$$x(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t)$$

$X(j\omega)$ $\quad H(j\omega)$ $\quad Y(j\omega) = X(j\omega)\, H(j\omega)$

⟹ eg. $\dfrac{d^2 y(t)}{dt^2} + 2\dfrac{dy(t)}{dt} = \dfrac{dx(t)}{dt} + 3\,x(t)$

Take F.T. of both sides:

$$(j\omega)^2\, Y(j\omega) + 2(j\omega)\, Y(j\omega) = j\omega\, X(j\omega) + 3\, X(j\omega)$$

$$Y(j\omega)\left((j\omega)^2 + 2j\omega\right) = X(j\omega)\left(j\omega + 3\right)$$

↪ $H(j\omega) = \dfrac{Y(j\omega)}{X(j\omega)} = \dfrac{j\omega+3}{(j\omega)^2 + 2j\omega} = \dfrac{N(j\omega)}{D(j\omega)}$ → polynomials in $j\omega$

Numerator: polynomial corresp. right hand side

Denominator: LHS

In general:

⟹

Linear constant coeff. ODEs:

General

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

Take F.T. of both sides

$$\Rightarrow \underbrace{H(j\omega)}_{\text{Freq. response}} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

I.F.T.

We can get the $h(t)$: impulse response of the system.

10) If $x(t)$ is real $\Rightarrow$ $X(j\omega)$ is conjugate symmetric:

$X(j\omega)$ (F.T) is a complex fn. of ω: $\therefore$ we talk about its polar representation (i.e. a magnitude & a phase) in complex plane

$$X(-j\omega) = X^*(j\omega)$$

$$\equiv X(j\omega) = X^*(-j\omega)$$

magnitude of F.T.

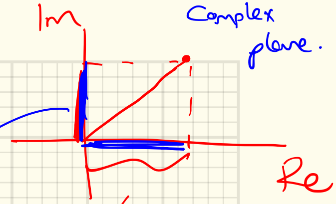
$$\Rightarrow |X(-j\omega)| = |X(j\omega)| \quad \text{even symmetry:}$$

phase of F.T.

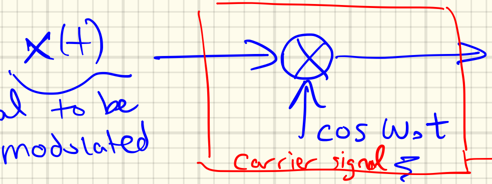
$$\measuredangle X(-j\omega) = -\measuredangle X(j\omega) \quad \text{odd symmetry.}$$

$$X(j\omega) = |X(j\omega)|\, e^{j\boxed{\angle X(j\omega)}}$$

$$= Re\{X(j\omega)\} + j \cdot Im\{X(j\omega)\}$$



11) Modulation Property:



modulation system:

$$y(t) = x(t) \cdot \cos \underline{\omega_0} t$$

signal that is modulated & transmitted over a communication channel.

carrier freq.

Given Input signal: $x(t) \longleftrightarrow X(j\omega)$

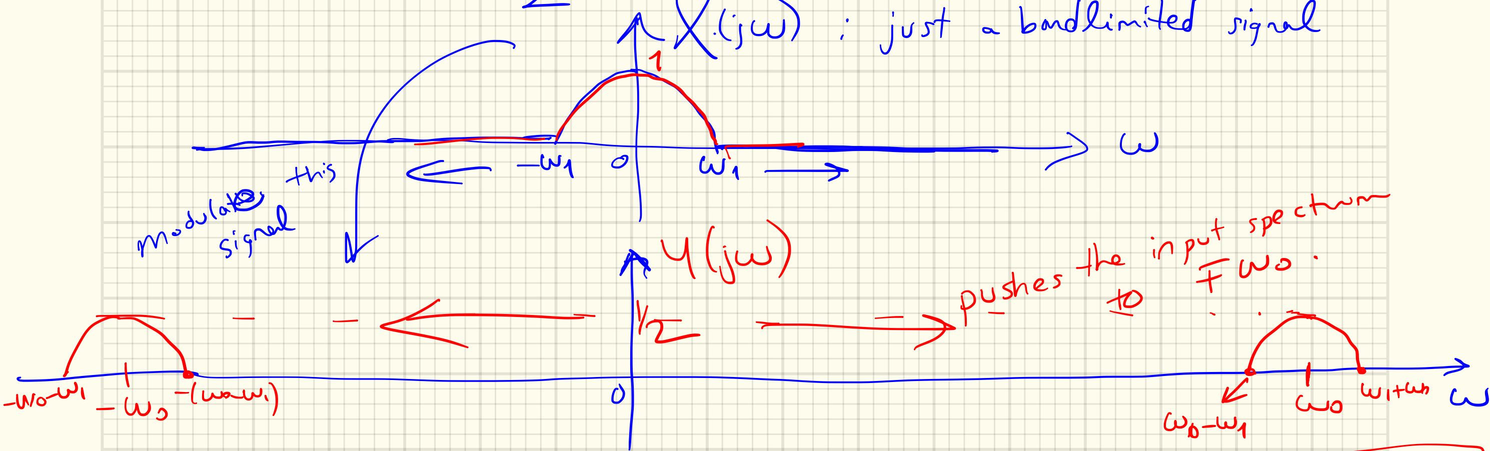
$y(t) \longleftrightarrow Y(j\omega) = ?$

Multiplication in time domain

$$Y(j\omega) = \frac{1}{2\cancel{\pi}}\left[X(j\omega) * \underbrace{F.T\{\cos\omega_0 t\}}_{\cancel{\pi}\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)}\right]$$

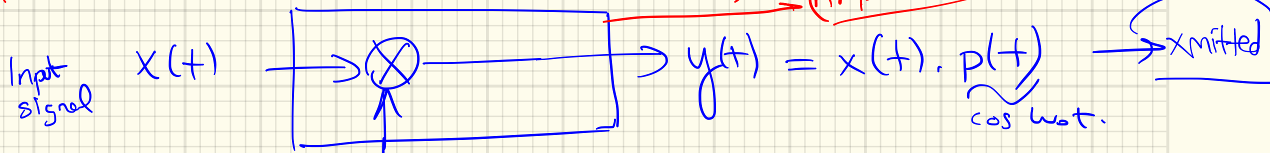
$$\rightarrow Y(j\omega) = \frac{1}{2}\left[X(j(\omega-\omega_0)) + X(j(\omega+\omega_0))\right]$$

$X(j\omega)$: just a bandlimited signal



Amplitude Modulation (12.2 first)

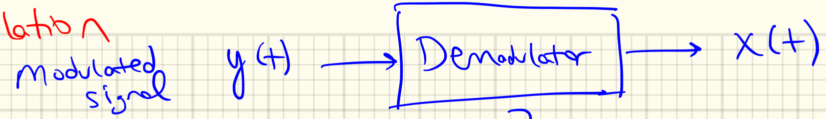
Amplitude Modulation System



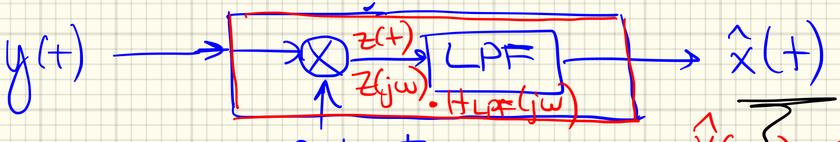
carrier signal $p(t) := \cos\omega_0 t$ w/ a carrier freq. ω_0.

Now given $\boxed{y(t)}$ → arrived at the receiver : Q: How to demodulate $y(t)$ to recover $x(t)$ back?

→ Demodulation

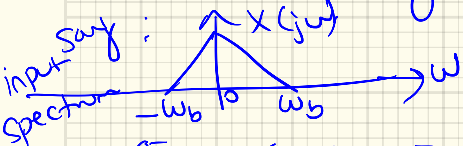


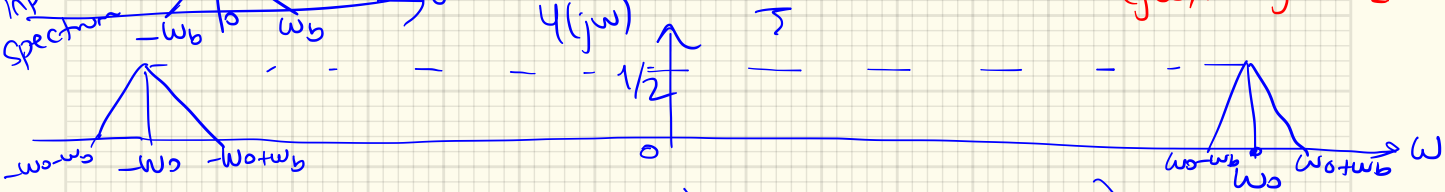
Demodulator System:



$$\hat{X}(j\omega) = Z(j\omega)\cdot H_{LPF}(j\omega)$$

say input spectrum:

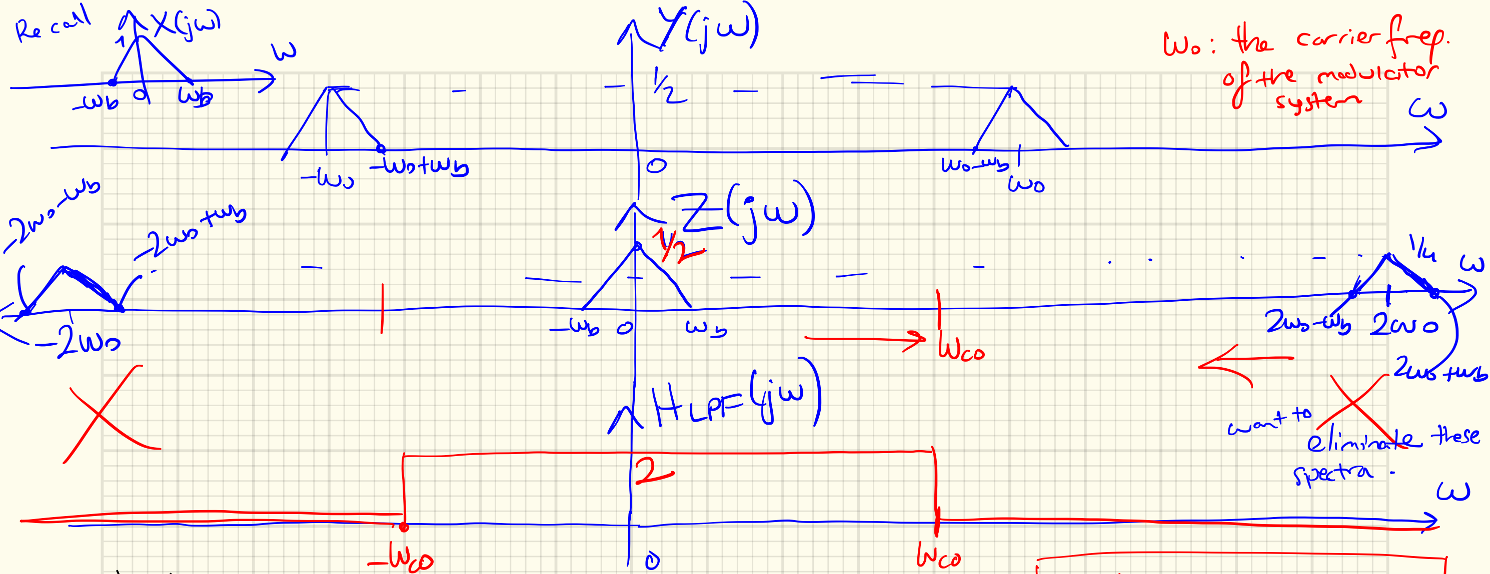




$$z(t) \rightarrow Z(j\omega) = \frac{1}{2}\left[Y(j(\omega-\omega_0)) + Y(j(\omega+\omega_0))\right]$$: using multiplication property.

$$Z(j\omega) = \frac{1}{2}\left[\frac{1}{2}X(j(\omega-2\omega_0)) + \frac{1}{2}X(j\omega) + \frac{1}{2}X(j\omega) + \frac{1}{2}X(j(\omega+2\omega_0))\right]$$

$$Z(j\omega) = \underbrace{\frac{1}{2}X(j\omega)}_{\text{desired spectrum}} + \underbrace{\frac{1}{4}X(j(\omega-2\omega_0)) + \frac{1}{4}X(j(\omega+2\omega_0))}_{\text{extra spectrum components I need to get rid of}}$$



Multiply the two spectrums

$Z(j\omega) \cdot H_{LPF}(j\omega) = \hat{X}(j\omega)$

ω_{co}: cut-off frequency of the LPF of the Demodulator system

$$\boxed{\begin{aligned} &\omega_{co} < 2\omega_0 - \omega_b \\ \& \ &\omega_{co} > \omega_b \end{aligned}}$$

→ $x(t)$ (ie the modulated & Xmitted signal from the transmitter) is recovered at the receiver

This is the idea behind Amplitude Modulation.

$$H_{LPF}(j\omega) = \begin{cases} 2, & \omega_b < |\omega| < 2\omega_0 - \omega_b \\ 0, & o/w \end{cases}$$

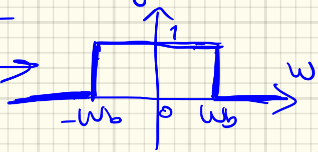
$$\hat{X}(j\omega) = H_{LPF}(j\omega) \cdot Z(j\omega) = X(j\omega) \checkmark$$

Now let's do solve some F.T. taking examples.

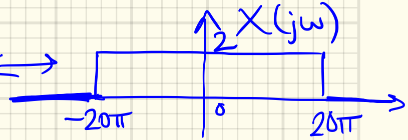
Ex: Given $x(t) = \dfrac{2\sin(20\pi t)}{\pi t}$, $y(t) = 5\dfrac{\sin(10\pi t)}{\pi t}$

Q: Find $z(t) = x(t) * y(t) = F.T^{-1}\{X(j\omega) \cdot Y(j\omega)\}$

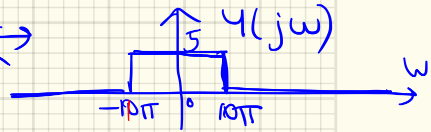
Recall (from the table) $\dfrac{\sin(\omega_b t)}{\pi t} \overset{F.T}{\longleftrightarrow}$



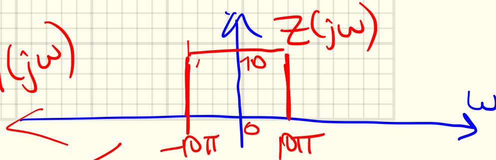
$\dfrac{2\sin(20\pi t)}{\pi t} \longleftrightarrow$



$\dfrac{5\sin(10\pi t)}{\pi t} \longleftrightarrow$



$Z(j\omega) = X(j\omega) \cdot Y(j\omega)$



$z(t) = F.T^{-1}\{Z(j\omega)\}$

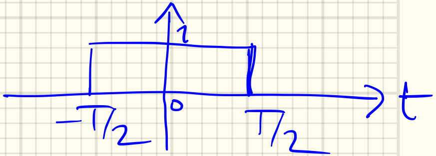
$z(t) = 10\dfrac{\sin(10\pi t)}{\pi t}$

Ex: $$X(j\omega) = 2\,\frac{\sin\left(3(\omega-2\pi)\right)}{(\omega-2\pi)} \longleftrightarrow F.T^{-1}\{\cdot\}$$

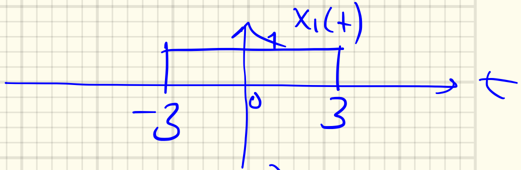
$x(t) = ?$

From Table 11.2

$$\frac{\sin \omega T/2}{\omega/2} \longleftrightarrow$$



$$\frac{\sin 3\omega}{\omega/2}$$

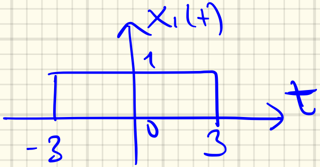


Do a freq. shift

$$e^{j\omega_0 t}\,x(t) \longleftrightarrow X(j(\omega-\omega_0))$$

$$e^{j2\pi t}\,x_1(t) \longleftrightarrow X_1(j(\omega-2\pi))$$

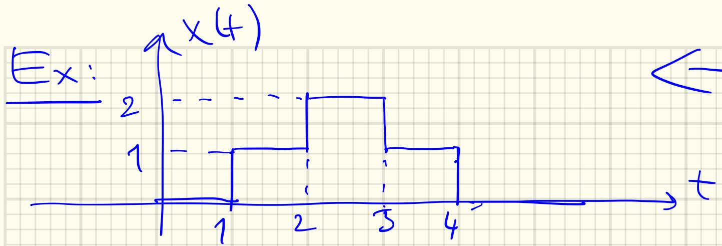
$e^{j2\pi t}\cdot$



$$\longleftrightarrow 2\,\frac{\sin\left(3(\omega-2\pi)\right)}{(\omega-2\pi)}$$

$x(t)$

Ex:

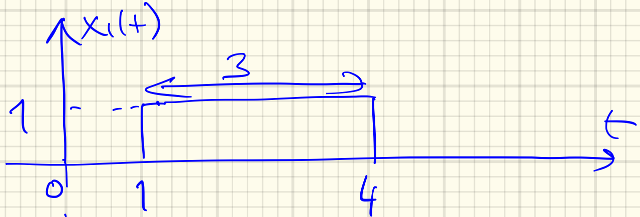


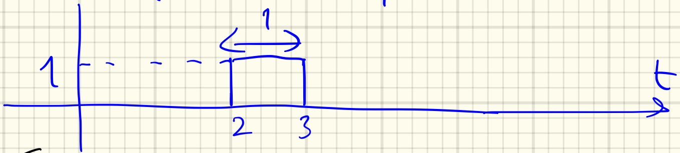
$$\longleftrightarrow \quad X(j\omega) = ?$$

Notice

$$x(t) = x_1(t) + x_2(t)$$

$$X(j\omega) = X_1(j\omega) + X_2(j\omega)$$





exercise: Calculate the result

to get:

$$X(j\omega) = e^{-j\omega 5/2}\left(\frac{\sin(3\omega/2)}{\omega/2} + \frac{\sin(\omega/2)}{\omega/2}\right)$$

Ex: $h(t) = 2e^{-2t}\,u(t) - e^{-t}\,u(t)$

$\Updownarrow$

$H(j\omega) = ?$ $\quad 2\dfrac{1}{(2+j\omega)} - \dfrac{1}{1+j\omega}$ $\quad (\star)$

If we were given $H(j\omega) = \dfrac{j\omega}{(2+j\omega)(1+j\omega)}$.

$FT^{-1}\{\cdot\}$ we should go back to $(\star)$ form to take $F.T^{-1}\{\cdot\}$

Partial Fraction Expansion: (PFE)

Given $H(j\omega) = \dfrac{N(j\omega)}{D(j\omega)} = \dfrac{j\omega}{(j\omega)^2 + 3j\omega + 2}$

eg. $s^2 + 3s + 2$

roots $s_{1,2} = -1, -2$

$$\frac{j\omega}{(j\omega+2)(j\omega+1)} = \frac{A}{(j\omega+2)} + \frac{B}{(j\omega+1)} \rightarrow \text{find } A \,\&\, B.$$

$$\left(\frac{j\omega\,\cancel{(j\omega+2)}}{\cancel{(j\omega+2)}(j\omega+1)} = \underline{A} + \underset{0}{\frac{B\cdot\cancel{(j\omega+2)}}{(j\omega+1)}}\right)\Bigg|_{j\omega=-2}$$

$$\frac{-2}{-1} = A \rightarrow A = 2$$

$$\frac{j\omega\,\cancel{(j\omega+1)}}{(j\omega+2)\cancel{(j\omega+1)}} = \cancel{\frac{A(j\omega+1)}{j\omega+2}} + B \Bigg|_{j\omega=-1}.$$

$$-\frac{1}{1} = B \rightarrow B = -1.$$

$$H(j\omega) = \frac{2}{j\omega+2} - \frac{1}{j\omega+1}$$

$$\rightarrow h(t) = 2e^{-2t}u(t) - e^{-t}u(t)$$

Ex: Given $$Y(j\omega) = \frac{1-\omega^2+j\omega}{2-\omega^2+j3\omega} \begin{array}{l} \rightarrow \text{degree of polynomial } 2 \\ \rightarrow \text{degree} = 2. \end{array}$$

★ Given $X(j\omega) = \dfrac{N(j\omega)}{D(j\omega)}$ rational form

If $\deg(N(j\omega)) < \deg(D(j\omega)) \rightarrow$ do PFE as we did in the previous example

degree " $\geq$ " $\rightarrow$ divide first to get $\deg N < \deg D$

$$\begin{array}{r|l} -\omega^2+j\omega+1 & -\omega^2+3j\omega+2 \\ \hline -\omega^2+3j\omega+2 & 1 \\ \hline -2j\omega-1 & \end{array}$$

$$\rightarrow Y(j\omega) = \underset{\sim}{1} + \frac{-j2\omega-1}{(j\omega)^2+3j\omega+2} \rightarrow \frac{s^2+3s+2}{(s+1)(s+2)}$$

do PFE

$$\frac{-j2\omega-1}{(j\omega+2)(j\omega+1)} = \frac{A}{j\omega+2} + \frac{B}{j\omega+1} \rightarrow \begin{array}{l} A=-3 \\ B=1 \end{array}$$

$$Y(j\omega) = 1 - \frac{3}{j\omega+2} + \frac{1}{j\omega+1}$$

I.F.T $\downarrow$

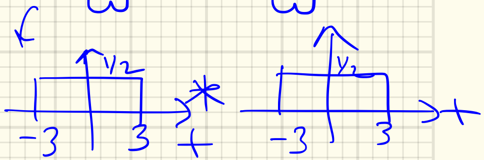
$$y(t) = \delta(t) - 3e^{-2t}u(t) + e^{-t}u(t)$$

Exercise:

$$H(j\omega) = \frac{\sin^2(3\omega)\cdot\cos(\omega)}{\omega^2} \Longleftrightarrow h(t) = ?$$

$$= \underbrace{\cos\omega}_{H_1(j\omega)} \underbrace{\left(\frac{\sin 3\omega}{\omega}\right)^2}_{H_2(j\omega)}$$

$$H_2(j\omega) = \frac{\sin 3\omega}{\omega}\cdot\frac{\sin 3\omega}{\omega}$$



$$h_1(t) = \frac{1}{2}\left(\delta(t+1) + \delta(t-1)\right)$$

$$h(t) = h_1(t) * h_2(t)$$

$$h(t) = \frac{1}{2}\left[h_2(t+1) + h_2(t-1)\right]$$

Next: Sampling in Freq. Domain $\Rightarrow$ Go to Slides

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(Under Sinif Dosyaları @ Ninova)