BLG 354 E Signals & Systems

08.03.2021

Q. What is a signal? A quantity that varies over time or space
1D, 2D, 3D, video, medical

A signal is a function of independent variable(s)
n-D. 3D, 4D..

1D fn. of time

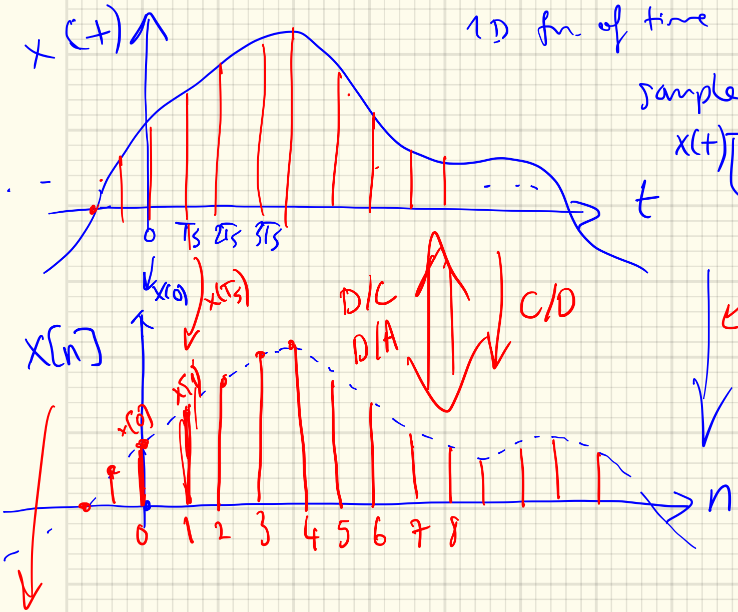
$I(x,y)$: 2D fn. of space
image fn.

sample every T_s seconds

$x(t)$ $\boxed{\text{continuous-time signal}}$

t : continuous var.
↓
n : integer index

$\boxed{x[n] : \text{discrete-time signal}}$



Systems

$x(t) \longrightarrow \boxed{C/D} \text{ converter} \longrightarrow x[n]$

Analog-to Digital

system

$x[n] \longrightarrow \boxed{D/C} \text{ converter} \longrightarrow x(t)$

Digital

$[\,0 \quad 1.2 \quad 2.3 \quad 4.2 \quad 5.6 \quad - \quad - \quad -\,]$

Array of numbers : Vectors_1D
matrices - 2D .
Tensors - nD .

Sinusoidal Signals:

$$x(t) = A \sin(\omega t + \phi)$$

Amplitude

Phase

$$\sin(\overbrace{\omega t}^{rad})$$

sec.

ω : radians/sec

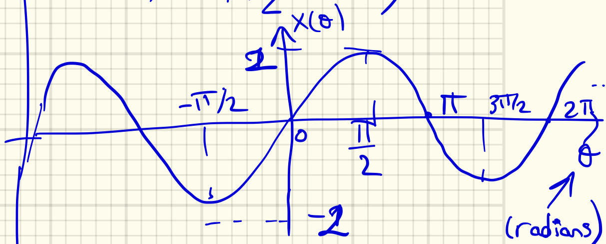
Angular frequency

ωt : radians.

$\omega = 2\pi f$: f : $\frac{1}{\sec}$: Hz

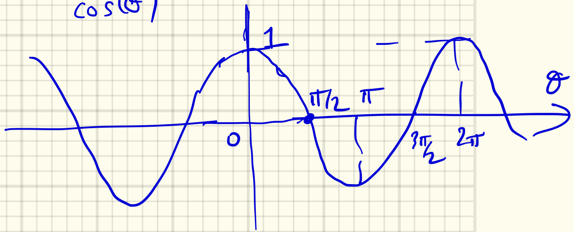
frequency

ex: $x(\theta) = 2\sin(\theta)$ $\theta \in (-\infty, \infty)$



$$\boxed{\sin(\theta) = \cos\left(\theta - \frac{\pi}{2}\right)}$$

Argument of sine & cosine is Radian.



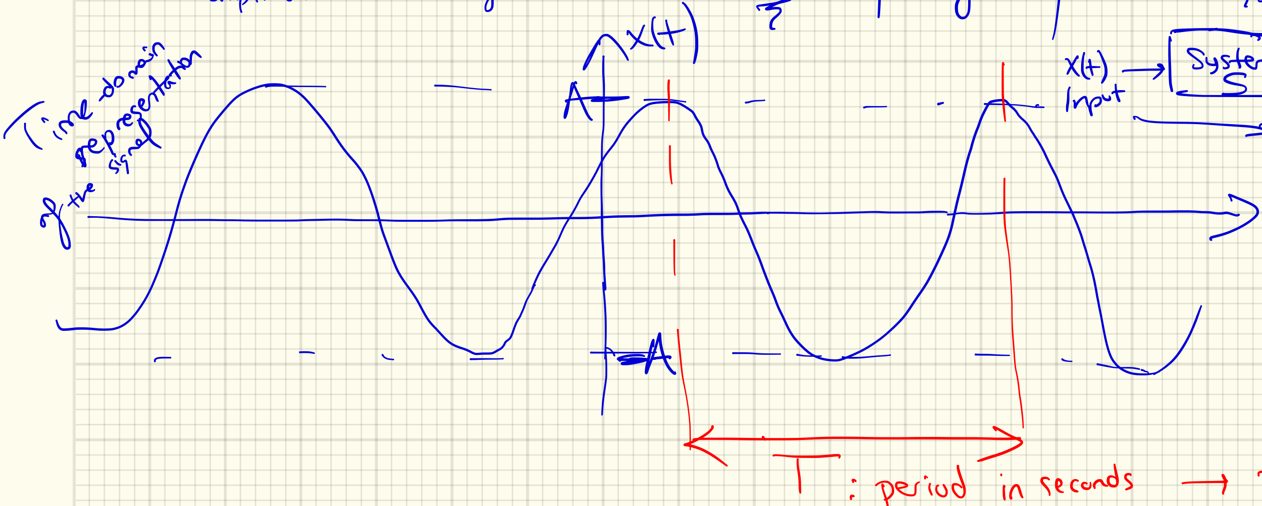
$x(t) = A\cos(\omega t + \phi)$

- A: amplitude
- $\omega = 2\pi f$ → f: frequency
- ϕ → phase: (radians)

Signal Processing
manipulating signals thru systems & algo.

$x(t)$ Input → System S → $y(t)$ output.

→ Fourier series
Fourier Xform



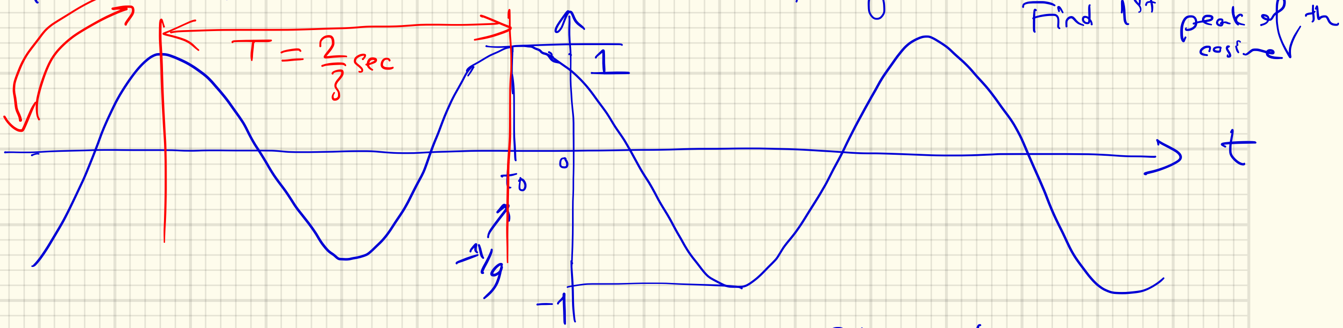
→ $f = \frac{1}{T}$ Hz

$\omega = \frac{2\pi}{T}$ rad.

Angular frequency → $\boxed{\omega = 2\pi f}$

Ex: $x(t) = \cos(3\pi t + \pi/3) \rightarrow$ Frequency ? , Phase ? Plot it

Find 1st peak of the cosine



Angular freq $\Rightarrow \omega = 3\pi$ rad/s

Phase $\Rightarrow \phi = \frac{\pi}{3}$ rad.

Freq $f = \frac{3\pi}{2\pi} = 1.5\,Hz \Rightarrow$ Period $T = \frac{2}{3}$ sec.

t_0: set the argument of cosine to 0 : $t_0 = \frac{-\pi/3}{3\pi} = -\frac{1}{9}$ sec.

Amplitude $A = 1$.

$\boxed{t_0 = -\frac{\phi}{\omega}} \rightarrow$ phase shift corresponds to a time shift

$A, \omega, (f), \phi$: parameters of the sine wave.

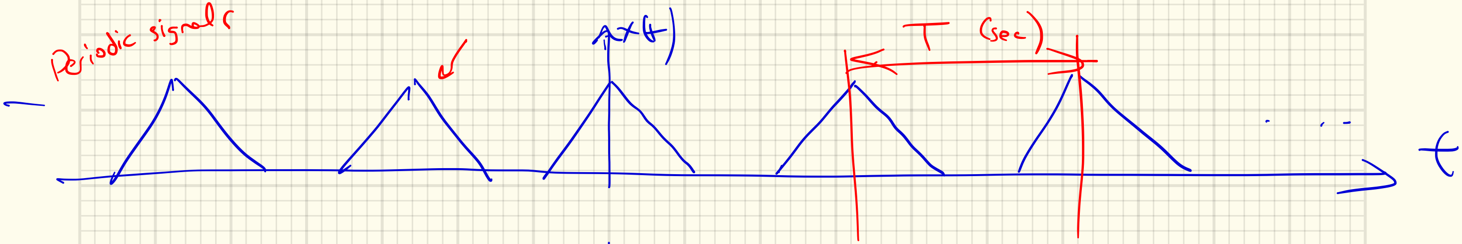
Periodic Signal :

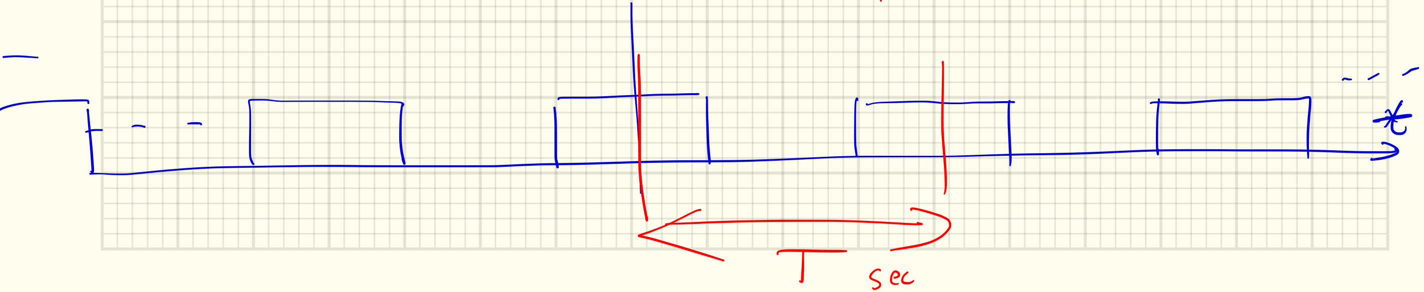
Def : If $x(t) = x(t+T)$ $\forall t$ then $x(t)$ is a periodic signal ; T is a period of $x(t)$

ex : $x(t) = 3 \sin\left(4\pi t + \frac{\pi}{8}\right) \rightarrow$ Find the period T .

— Sinusoids are periodic signals

Periodic signals

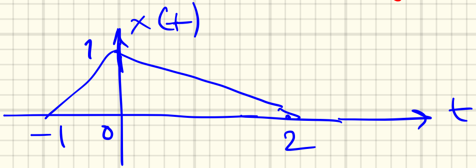


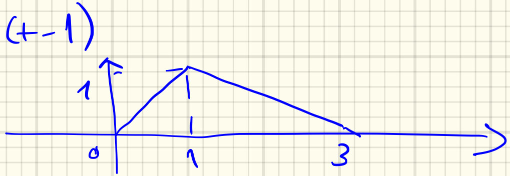


Shifting & Time Scaling of Signals

Shifting

eg.





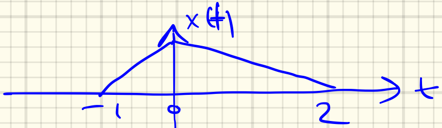
$t_0 > 0$

$y(t) = x(t - t_0)$: moves $x(t)$ to the right by t_0 sec.

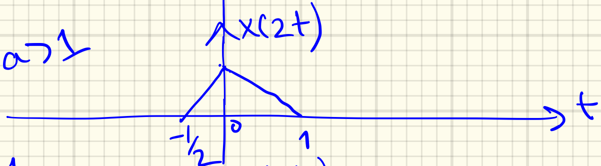
$y(t) = x(t + t_0)$ " " left " "

Time Scaling

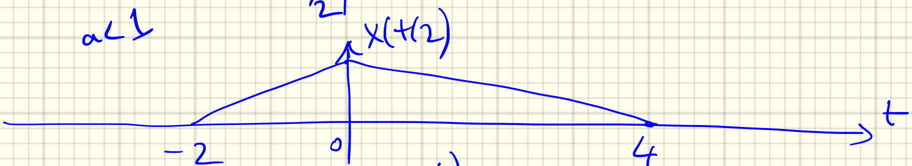
$x(at)$



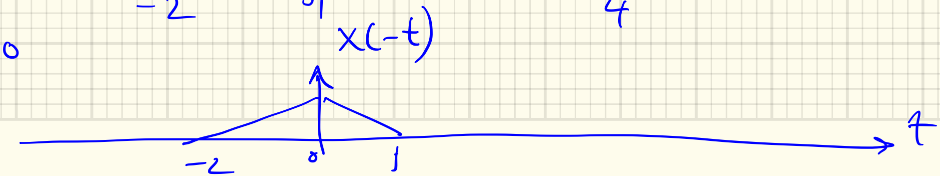
$a > 1$



$a < 1$



$a < 0$

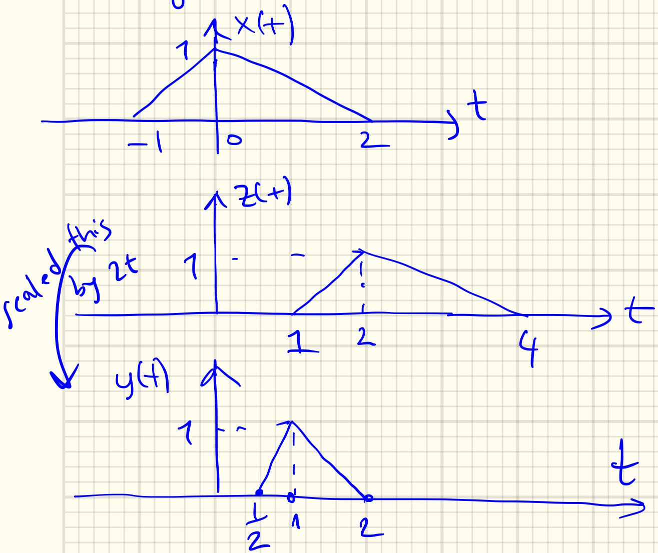


Both Time Shift & Scale : Rule of thumb :

☆ First SHIFT then SCALE.

$y(t) = x(at + t_0)$

Ex: $y(t) = x(2t - 2)$



$z(t) = x(t - 2)$ first shift

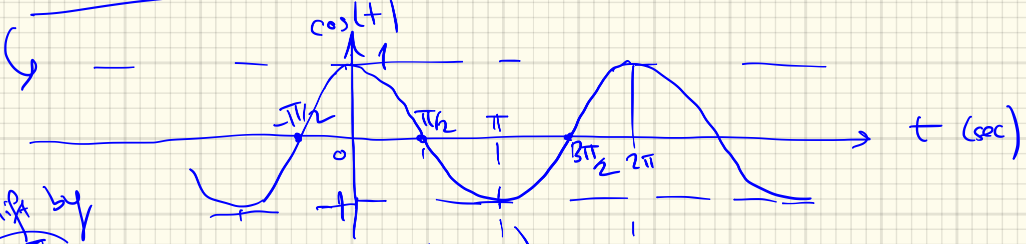
$y(t) = z(2t)$

$y(t) = x(2t - 2)$

Do the exercise :

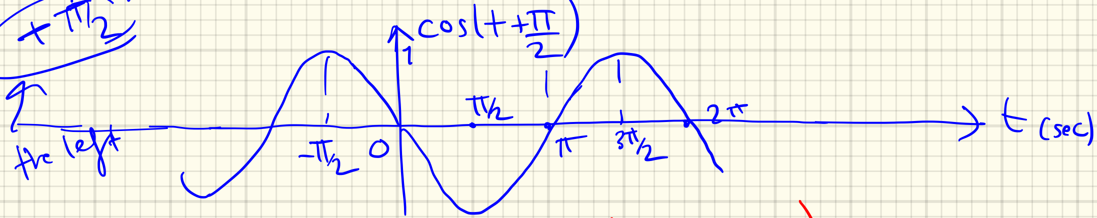
Exercise: $y(t) = \cos(2\pi t + \pi/2)$

Start with $x(t) = \cos(t)$ obtain.

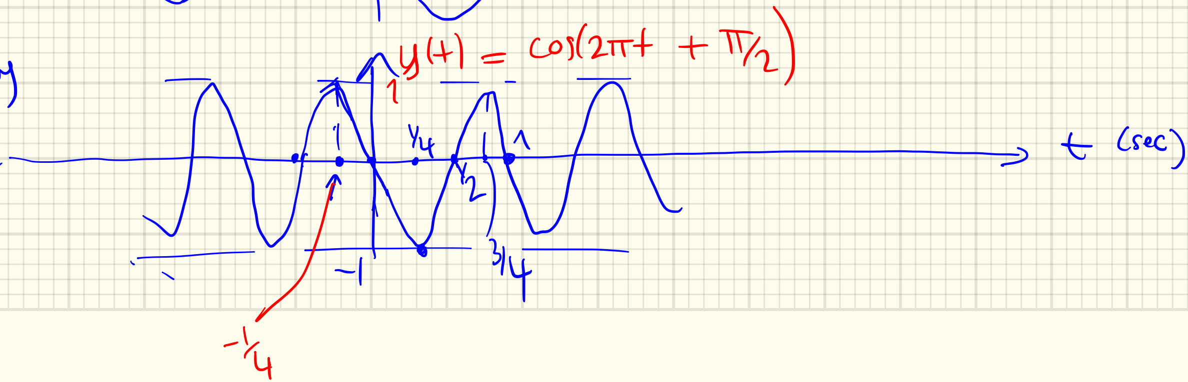


First shift by $+\pi/2$

Shift to the left



Scale by 2π.



Do this

↳ Exercise: Plot $x(t) = 7\cos(0.2\pi t + 0.8\pi)$ $\mp 2\pi$ → same signal

Phase is ambigous.

Find
$$\left\{ \begin{array}{llll} A, & \omega & & \\ & f, & T, & t_0 : \text{1st peak time point} \\ & \phi & & ? \end{array} \right.$$

Given a sine /cosine plot:

You should be able to write down these parameters of the signal. ⟶ mathematical expression.

real signal

$$\boxed{A\cos(\omega t + \phi)}$$

→ Go to complex sinusoids

Sinusoids $\longrightarrow x(t) = A e^{j(\omega t + \phi)}$: Complex Sinusoids

Complex number / plane will represent sinusoids.

Basis functions

$$x[n] = \underbrace{\sum_{k=-\infty}^{\infty} \psi_k[n]}_{} \cdot a_k$$

signal

coeff.

ψ: basis functions

ψ: in this course

Fourier basis fn.s.

$\equiv$ complex sinusoids

Recall: Complex Plane

extension of real numbers

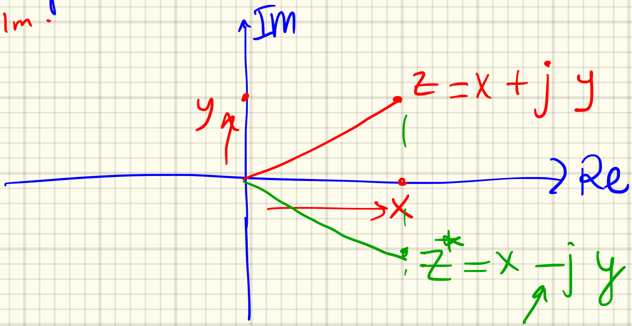
$z = x + jy$

Re

Im.

imag. number

$z \longrightarrow$ complex plane



j:

$\boxed{j^2 = -1}$

$j^3 = -j$

$j^4 = 1$

$j^5 = j$

$\vdots$

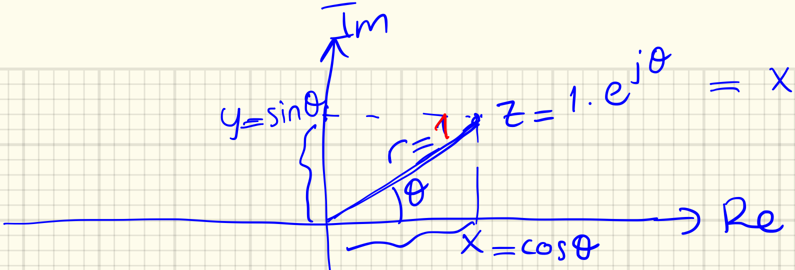
$z = x + jy$: Cartesian form of the complex number

Polar form : $z = re^{j\theta}$ $\qquad r = \sqrt{x^2 + y^2}$

$\longrightarrow$

$r=\sqrt{x^2+y^2}$

$\theta = \mathrm{atan}\left(\frac{y}{x}\right)$



$z = 1\cdot e^{j\theta} = x + jy$

Know this very well.

Euler's formula $\Rightarrow$ $\boxed{z = re^{j\theta} = r\cos\theta + rj\sin\theta}$

Real=0 Img=2.

$z = x + jy \Rightarrow z = 0 + j\cdot 2$.

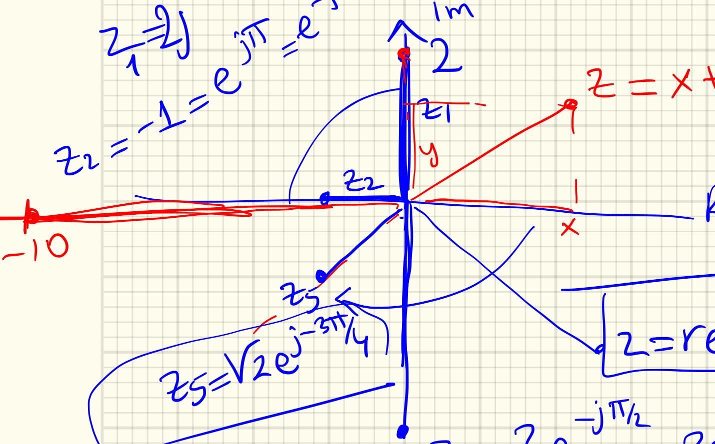
$z = 0 - j\cdot 3$

$z = -10 + j\cancel{0}$

$z = 1$

$z_1 = 2j$

$z_2 = -1 = e^{j\pi} = e^{-j\pi}$



$\boxed{z = re^{j\theta}}$

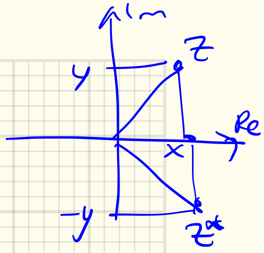
$z_5 = \sqrt{2}e^{j-3\pi/4}$

$z_3 = 3e^{-j\pi/2} = 3e^{j3\pi/2} \ldots$

$\boxed{z_5 = -1 - j}$

$r = \sqrt{2}$, $\theta = -\frac{3\pi}{4} = \mathrm{atan}\left(\frac{-1}{-1}\right)$

$$z = Re\{z\} + j\,Im\{z\} \rightarrow \boxed{Re\{z\} = \frac{z+z^*}{2}} \qquad Im\{z\} = \frac{z-z^*}{2j}$$



Euler formula: $r=1$

$$\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$$

2 variants

$$\boxed{\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}} \qquad \boxed{\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}}$$

Know these

writing a real sinusoid ito 2 complex exponentials (or complex sinusoids)

Complex Sinusoids:

$$x(t) = A e^{j(\omega_0 t+\phi)} = \boxed{Ae^{j\phi}}\, e^{j\omega_0 t}$$

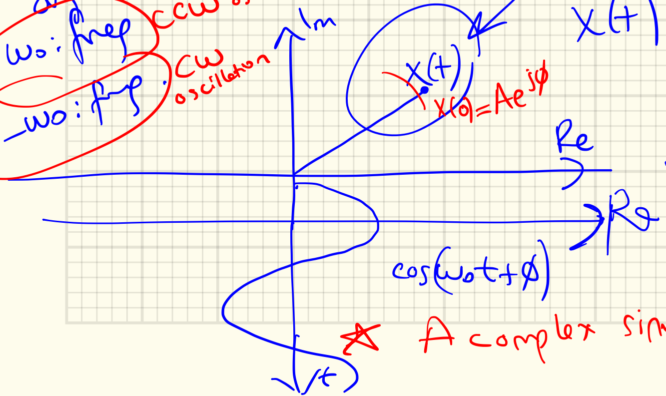
phasor

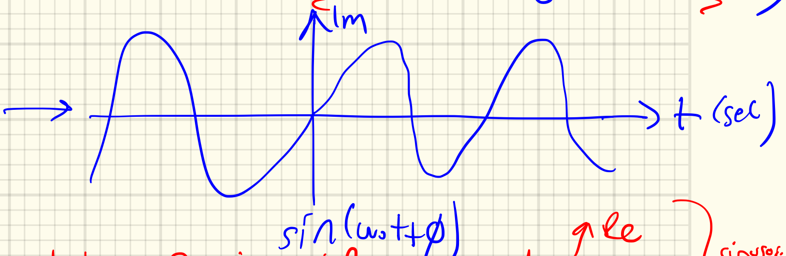
$$x(t) = A\cos(\omega_0 t+\phi) + jA\sin(\omega_0 t+\phi)$$

angular

ω_0: freq ccw oscillation

$-\omega_0$: freq cw oscillation





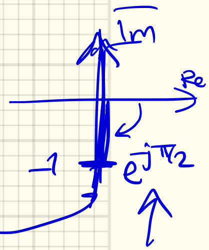
★ A complex sinusoid has 2 sinusoidal components → Re, → Imag. } sinusoi.

Write it ito a complex sinusoid :

$$\rightarrow A\cos(\omega_0 t+\phi) = \mathrm{Re}\{A e^{j(\omega_0 t+\phi)}\}$$

$$A\sin(\omega_0 t+\phi) = \mathrm{Im}\{A e^{j\omega_0 t+\phi)}\}$$

ex : Evaluate $x(t) = \mathrm{Re}\{-3j\, e^{j\omega t}\}$: $-j =$



$$= \mathrm{Re}\{3\, e^{-j\pi/2} e^{j\omega t}\}$$

$$x(t) = 3\cos(\omega t - \tfrac{\pi}{2})$$

Recall ex : $\cos(a+b) = ?$ Use Euler formula

$$= \mathrm{Re}\{e^{j(a+b)}\} = \mathrm{Re}\{e^{ja}\cdot e^{jb}\}$$

$$= \mathrm{Re}\{(\cos a + j\sin a)\cdot(\cos b + j\sin b)\}$$

$$= \mathrm{Re}\{\cos a\cos b + \underbrace{j\cos a\sin b + j\sin a\cos b}\, + j^2 \sin a\sin b\}$$

$$\cos(a+b) \rightarrow \cos a\cos b - \sin a\sin b$$

~~exercise:~~ $\sin(2a) = \mathrm{Im}\{e^{j2a}\} = \mathrm{Im}\{e^{ja} \cdot e^{ja}\}$

Use Euler formula

$\vdots$

$= 2\cos a \sin a$.

Adding Sinusoidal Signals w/ same Frequency Using Phasors

$$x(t) = \underbrace{\boxed{A e^{j\phi}}}_{\text{Phasor}} e^{j\omega_0 t}$$

ω_0: frequency

Ex:

$$x_1(t) = 3\cos\left(2\pi t + \frac{\pi}{4}\right)$$

$$x_2(t) = 4\cos\left(2\pi t - \frac{\pi}{3}\right)$$

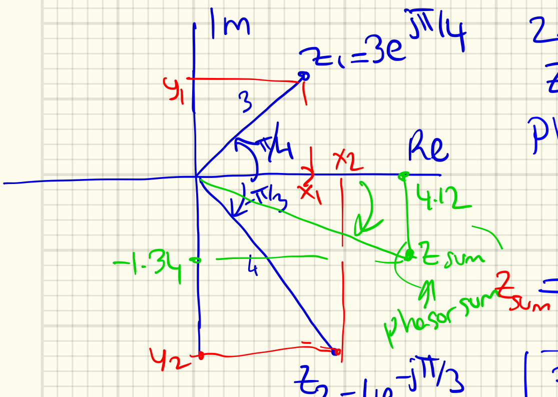
$$x_1(t) + x_2(t) = \ ? = x(t) \quad \text{sum signal}$$

Same freq $\omega_0 = 2\pi$ rad/s.

$f = 1$ Hz.

Go to Phasor Representation to do the addition:

$$x(t) = \mathrm{Re}\left\{3e^{j(2\pi t + \pi/4)}\right\} + \mathrm{Re}\left\{4e^{j(2\pi t - \pi/3)}\right\} = \mathrm{Re}\left\{e^{j2\pi t}\left\{3e^{j\frac{\pi}{4}} + 4e^{-j\frac{\pi}{3}}\right\}\right\}$$



$$\left.\begin{aligned} z_1 &= 3e^{j\pi/4} \\ z_2 &= 4e^{-j\pi/3} \end{aligned}\right\} \text{Addition of 2 phasors}$$

Phasor sum:

$$3\left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right) + 4\cos\left(-\frac{\pi}{3}\right) + j4\sin\left(-\frac{\pi}{3}\right)$$

$$z_{sum} = 3\cdot\frac{1}{\sqrt{2}} + j3\frac{1}{\sqrt{2}} + 4\cdot\frac{1}{2} - j\cdot 4\frac{\sqrt{3}}{2} \cong \boxed{4.12 - j1.34}$$

$$r = \sqrt{(4.12)^2 + 1.34^2}$$

$$\boxed{\theta = \mathrm{atan}\left(\frac{-1.34}{4.12}\right)} \cong -\frac{\pi}{10}$$

$\pi \to 3.14\ldots$ $= 0.314$

$$\boxed{z_{sum} \cong 4.3\, e^{-j\pi/10}}$$

sum signal

$$x(t) = \mathrm{Re}\left\{ 4.3\, e^{-j\pi/10} \cdot e^{j2\pi t} \right\}$$

$$\underbrace{x(t)}_{\text{Resulting signal}} = \underbrace{4.3}_{\text{different amplitude}} \cos(\underbrace{2\pi t}_{\text{same freq.}} \underbrace{- \frac{\pi}{10}}_{\text{different phase}})$$

Reading Assignment

Chapter 2

SignalProcFirst

In general:

$$\sum_{k=1}^{N} A_k \cos(\underbrace{\omega_0 t}_{\text{same freq.}} + \phi_k) = A \cos(\omega_0 t + \phi)$$

How to find A, ϕ?

Later: $\cos(k \cdot \omega_0 t + \phi_k)$

$$\mathrm{Re}\left\{ \sum_{k=1}^{N} A_k e^{j(\omega_0 t + \phi_k)} \right\} = \mathrm{Re}\left\{ \underbrace{\left(\sum_{k} A_k e^{j\phi_k} \right)}_{A_1 e^{j\phi_1} + A_2 e^{j\phi_2} + \dots + A_k e^{j\phi_k}} \cdot e^{j\omega_0 t} \right\}$$

$$= \mathrm{Re}\left\{ A e^{j\phi} \cdot e^{j\omega_0 t} \right\}$$

$$= A \cos(\omega_0 t + \phi)$$

Do this at home:

Exercise: $x(t) = 2\cos\left(300\pi t + \frac{3\pi}{4}\right) + 2\sqrt{2}\cos\left(300\pi t + 0{,}005\right)$

Solution: you should find: $x(t) = 2\cos\left(300\pi t + \frac{5\pi}{4}\right)$