BLG 354E Signals & Systems

Spring 2021

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Last time:

— Sinusoidals (signals) are very important in signal processing

→ All signals (data) have frequency content.

real sinusoidal $x(t) = A \cos(\underbrace{\omega t + \phi}_{\theta})$

amplitude: A; $\omega = 2\pi f$ (rad/s), f (1/s); phase: ϕ

→ $x(t) = \int c_k \psi_k(t)\,dt$

real basis functions: $\psi_k(t)$

c_k: $\langle x(t), \psi_k(t) \rangle$

basis fn

complex sinusoidals

$x(t) = A e^{j(\omega t + \phi)}$

$= A\cos(\omega t + \phi) + jA\sin(\omega t + \phi)$: Euler formula

→ Chapter 3

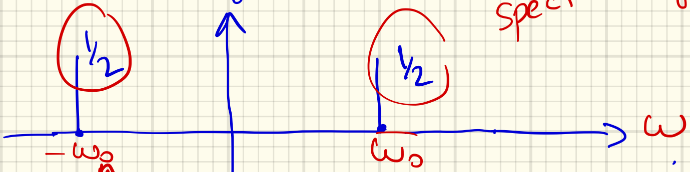
Chapter 3 Spectrum Representation

Def: (Spectrum Representation): A compact representation of "frequency content" of a signal.

Given a signal

$$x(t) = \cos(\underbrace{2\pi f_0}_{\omega_0} t) \rightarrow \text{e.g.}$$

spectrum of $x(t)$



ω (rad/s)

(could be f)

$$x(t) = \cos(\omega_0 t) = \frac{1}{2} e^{+j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t}$$

(inverse Euler)

existing frequencies in the signal.

Ex: $x(t) = \underbrace{3\cos\left(4\pi t + \frac{\pi}{3}\right)}_{1} + \underbrace{4\cos(6\pi t)}_{2}$

$$\underbrace{e^{j\omega_0 t}}_{\text{basis fn.}}$$

Q. Find the spectrum representation of $x(t)$

1) $3\cos(4\pi t + \frac{\pi}{3}) = \boxed{\frac{3}{2} e^{j\pi/3}} \, e^{j4\pi t} + \boxed{\frac{3}{2} e^{-j\pi/3}} \, e^{-j4\pi t}$

phasor — complex amplitudes — phasor

$\omega_1 = 4\pi$ rad/s, $\omega_{-1} = -4\pi$ rad/s

$f_1 = 2$ Hz, $f_{-1} = -2$ Hz.

$\phi_1 = \frac{\pi}{3}$, $\phi_{-1} = -\pi/3$

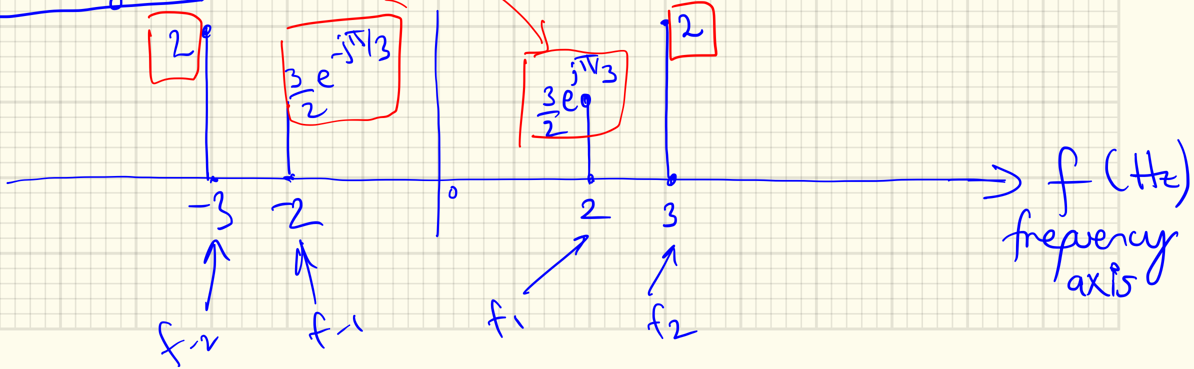
2) $\underbrace{4\cos(6\pi t)}$ $= \boxed{2} \, e^{j6\pi t} + \boxed{2} \, e^{-6\pi t}$

$f_2 = 3$ Hz, $f_{-2} = -3$ Hz

$\phi = 0$

x(t) has frequencies: $f_1 = 2$ Hz, $f_{-1} = -2$ Hz, $f_2 = 3$ Hz, $f_{-2} = -3$ Hz

Q: Plot the spectrum of x(t)



2_sided spectrum (for real signals, we have 2_sided spectrum)

symmetric w.r.t. y axis.

$$x(t) = 5\sin\left(7\pi t + \frac{\pi}{4}\right) \Rightarrow \text{Spectrum of } x(t)\,?$$

real signal

$\omega_1 = 7\pi$, $\omega_{-1} = -7\pi$ rad/s

$$= \boxed{\frac{5}{2j} e^{j\pi/4}} \cdot \boxed{e^{j\underbrace{7\pi}_{7\pi} t}} - \boxed{\frac{5}{2j} e^{-j\pi/4}} \cdot \boxed{e^{-j\underbrace{7\pi}_{} t}}$$

-7π

$j = e^{j\pi/2}$

phasor 1 $\Rightarrow \frac{5}{2} e^{-j\pi/2} \cdot e^{j\pi/4} = \boxed{\frac{5}{2} e^{-j\pi/4}}$

C_1

$\boxed{\text{phasor } 2}$ $= \frac{5}{2} \underbrace{e^{j\pi}}_{(-1)} \cdot \overbrace{e^{-j\pi/2}}^{e^{j\pi/2}} \cdot e^{-j\pi/4} = \boxed{\frac{5}{2} e^{j\pi/4}}$

$C_{-1} = C_1^{*}$

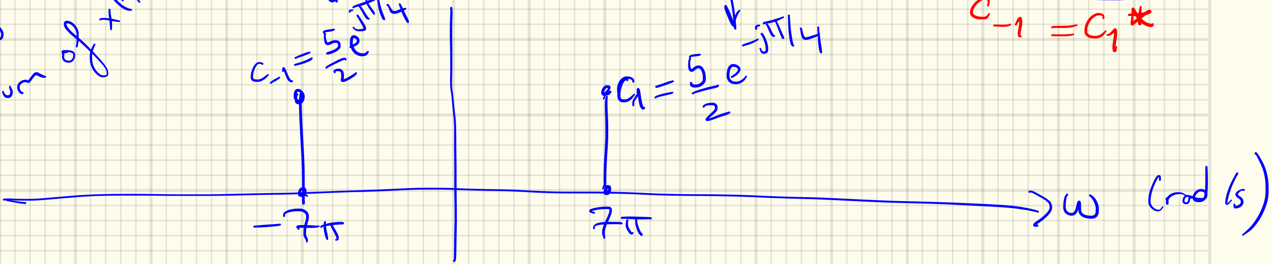
complex calculus

$-1 = e^{j\pi}$

$j = e^{j\pi/2}$

$-j = e^{-j\pi/2}$

2-sided spectrum of $x(t)$:



eg. plot spectrum of $\cos(2\pi t - \pi/7) \longrightarrow$

Ex: $x(t) = \underbrace{10}_{\text{constant part signal}} + 8\cos\left(2\pi(250)t + \frac{\pi}{2}\right) + 14\sin\left(2\pi(100)t + \frac{\pi}{6}\right)$

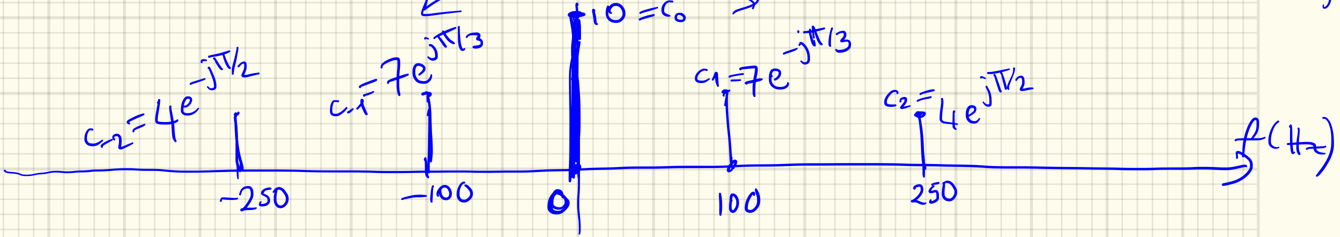
$\rightarrow f = \mp 250\text{Hz}$

$\rightarrow f = \mp 100\text{ Hz}$

$10e^{j0} \Rightarrow \omega = 0, \; f = 0$

* Constant signals have ZERO frequency. (in EE DC component)

complex conjugates of each other: b/c we have real signal



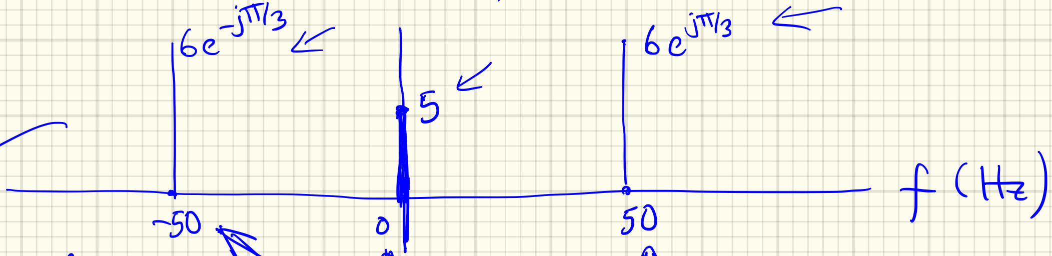
exercise: derive the coefficients in the spectrum: c_0, c_1, c_{-1}, c_2, c_{-2}

Note: Magnitude of the coefficients (phasor $Ae^{j\phi}$) should be positive.

$-5e^{j\pi/3} = \underbrace{e^{j\pi}}_{e^{-j\pi}} 5e^{j\pi/3} = 5e^{j4\pi/3} = e^{-j2\pi/3}$; $\mp k2\pi$; the same signal

$\sin(\omega t) = \cos\left(\omega t - \frac{\pi}{2}\right)$

Exercise: Give a signal's spectrum:



What is the signal x(t)?

$$x(t) = 5 + 12\cos\left(100\pi t + \frac{\pi}{3}\right)$$

exercise

$$\longrightarrow x(t) = ?\ A\sin(\omega t + \emptyset)$$

Multiplying Two Sinusoids:

Ex: $x(t) = \cos(\pi t) \cdot \sin(10\pi t)$ ← multiplication

$\omega = \pi$ rad/s, $f = 0.5$ Hz $\omega = 10\pi$ rad/s

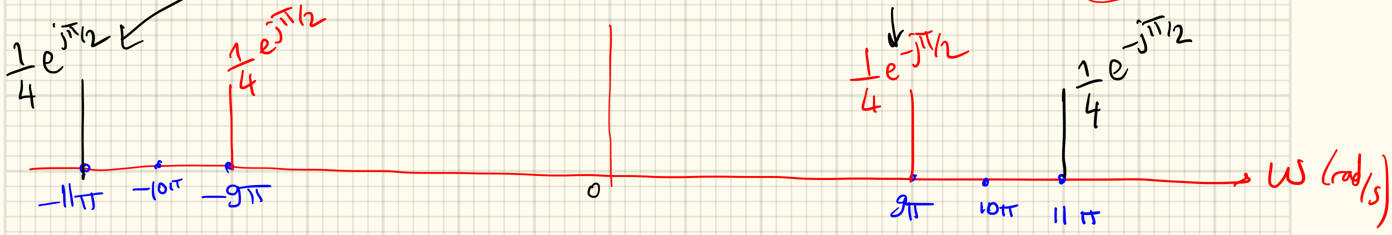
Find the spectrum

$$x(t) = \left(\frac{1}{2}e^{j\pi t} + \frac{1}{2}e^{-j\pi t}\right)\cdot\left(\frac{1}{2j}e^{j10\pi t} - \frac{1}{2j}e^{-j10\pi t}\right)$$

$$x(t) = \frac{1}{4j}e^{j11\pi t} - \frac{1}{4j}e^{-j9\pi t} + \frac{1}{4j}e^{j9\pi t} - \frac{1}{4j}e^{-j11\pi t}$$

Note: $\frac{1}{j} = e^{-j\pi/2}$, $-\frac{1}{j} = e^{j\pi/2}$

$$x(t) = \frac{1}{4}e^{-j\pi/2}\cdot e^{+j11\pi t} + \frac{1}{4}e^{j\pi/2}\cdot e^{-j11\pi t} + \frac{1}{4}e^{j\pi/2}\cdot e^{-j9\pi t} + \frac{1}{4}e^{-j\pi/2}\cdot e^{+j9\pi t}$$



from

$$x(t) = \frac{1}{2}\underbrace{\cos\left(9\pi t - \frac{\pi}{2}\right)}_{\sin(9\pi t)} + \frac{1}{2}\underbrace{\cos\left(11\pi t - \frac{\pi}{2}\right)}_{\sin(11\pi t)}$$

In general: $x(t) = \cos(\omega_{small}\, t + \cancel{\phi}) \cdot \cos(\omega_{big}\, t)$

Exercise

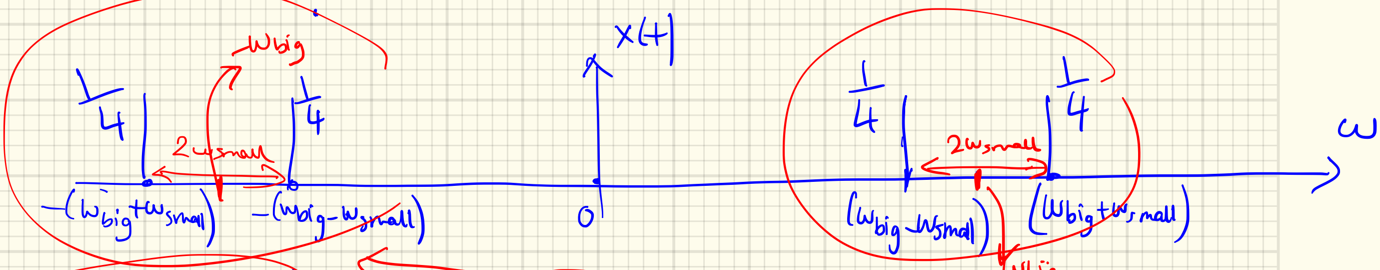
Show:

$$x(t) = \frac{1}{2}\cos\left((\omega_{big}+\omega_{small})\,t\right) + \frac{1}{2}\cos\left((\omega_{big}-\omega_{small})\,t\right)$$

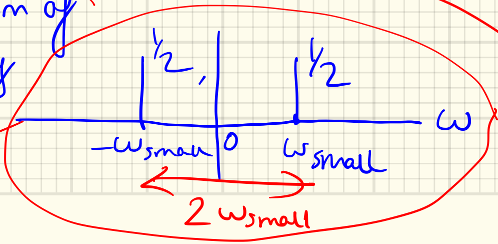
Start w/ $x(t) = \left(\frac{1}{2}e^{j\omega_{small}t} + \frac{1}{2}e^{-j\omega_{small}t}\right)\left(\frac{1}{2}e^{j\omega_{big}t} + \frac{1}{2}e^{-j\omega_{big}t}\right)$

continue

⋮



Note: The Spectrum of the low frequency signal



is carried to high frequencies

☆ This is the idea behind Amplitude modulation

In general $x(t) = v(t) \cdot \cos 2\pi f_c t$

$v(t)$: low freq. signal (eg. audio / speech signal) eg. 5kHz.

$\cos 2\pi f_c t$: carrier signal

f_c: carrier frequency eg. 1 MHz.

This is called modulating $v(t)$ w/ a carrier frequency f_c.

Ex: Given $v(t) = 5 + 4\cos(1000\pi t)$ → Low freq. signal compared to a carrier signal.

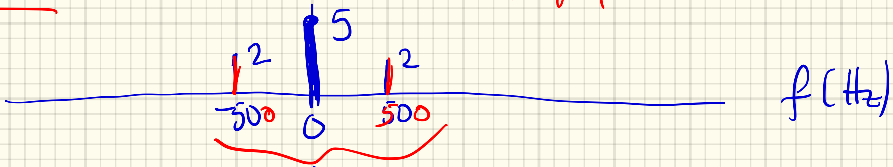
$f=0 \Rightarrow$ DC (constant) $\qquad f_1 = 500$ Hz

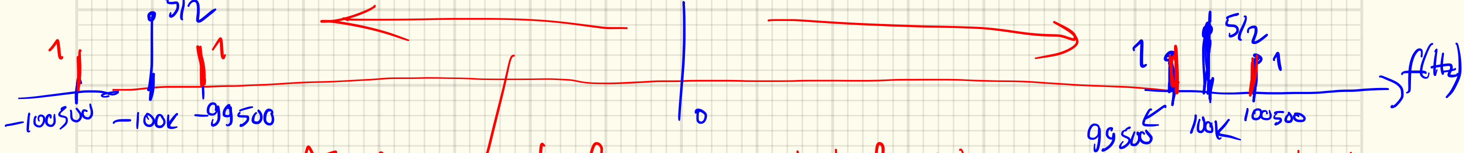
$x(t) = v(t) \cdot \cos(200000\pi t)$ → $f_c = 100$kHz

Spectrum of $x(t)$?

exercise: Derive this using Euler's formula.

$v(t)$ spectrum: low freq.





★ We carried frequency content of $v(t)$ to around 100kHz by MODULATION

BEAT NOTES: (≡ multiplying of 2 sinusoids) → 1 of them slowly-varying (≡ low freq)

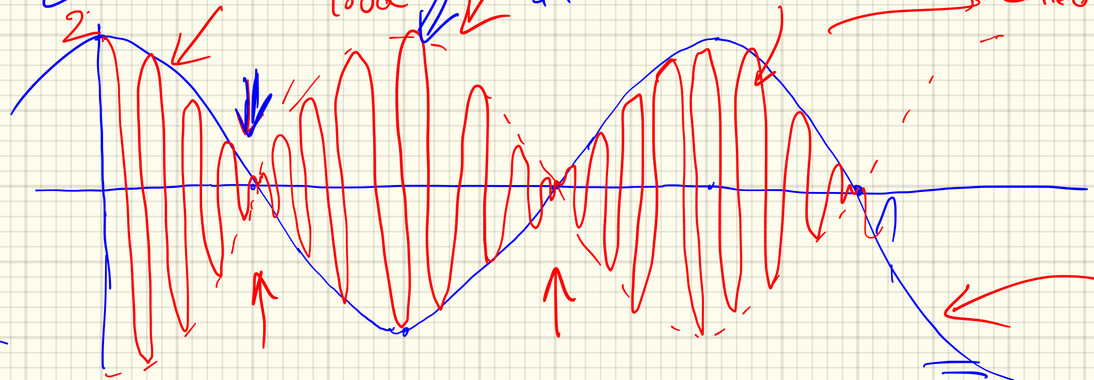
or Addition

other is rapidly-varying (high freq).

Ex: $$x(t) = 2\cos(2\pi \underbrace{20}_{\text{slowly-varying}} t) \cdot \cos(2\pi \underbrace{200}_{\text{rapidly varying}} t)$$

$\left(T = \frac{1}{20}\text{ sec}\right)$ rapidly varying $\left(T = \frac{1}{200}\text{ sec}\right)$

look at it in time



called "beating" phenomenon.

e.g. 2 Hz

Beat Note: Adding 2 sinusoids of slightly different frequencies

$$\cos(2\pi f_1 t)\cdot\cos(2\pi f_2 t) = \frac{1}{2}\cos(2\pi(f_1+f_2)t) + \cos(2\pi(f_1-f_2)t)$$

$f_1 = 222$ Hz — large

$f_2 = 2$ Hz. — small

$222 + 2$ → $\boxed{224 \text{ Hz.}}$

$222 - 2$ → $\boxed{220 \text{ Hz.}}$

Check & find the exact parameters.

2 frequencies that are slightly different are added.

Multiplying 2 sinusoids $\Longleftrightarrow$ Addition of 2 sinusoids

$\Rightarrow$ In general:

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \phi_k)$$

f_k:

which frequencies exist in this signal?

for each f_k: $\frac{1}{2} A_k e^{+j\phi_k}$ $\Leftarrow$ phasor coefficients.

" " $-f_k$: $\frac{1}{2} A_k e^{-j\phi_k}$ $\Leftarrow$ " .

f_0 : A_0

$k=0$: $a_0 = A_0$

$k=1\ldots N$ $a_k = \frac{1}{2} A_k e^{j\phi_k}$

$a_{-k} = \frac{1}{2} A_k e^{-j\phi_k}$

$$X(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

a_k, f_k pairs $\rightarrow$ form the spectrum of $x(t)$.

→ Concept of Fundamental Frequency:

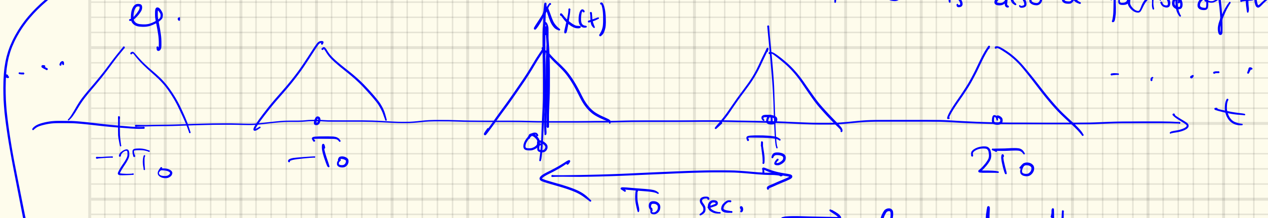
Recall Periodic Signal w/ period T_0 : $x(t+T_0) = x(t)$, $\forall t$

if $\exists\, T_0$ s.t → $x(t + kT_0) = x(t)$, $\forall t$

k integer.

smallest T_0 : fundamental period (kT_0 is also a period of the signal).

e.g.



→ $f_0 = \dfrac{1}{T_0}$ Hz.

Fundamental Frequency f_0:

We have a sum of sinusoidal signals:

$$x(t) = \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \phi_k)$$

☆ If $x(t)$ is periodic ; → $x(t) = \sum_k A_k \cos(2\pi k f_0 t + \phi_k)$

Finding f_0 is important!

$\Rightarrow$ If we can find an f_0 s.t. $f_0 = \underbrace{gcd}_{\text{greatest common divisor}}(f_k)$ where $\left(\frac{f_k}{f_0}\right)$ is an integer

then the signal $x(t)$ is $\boxed{\text{periodic}}$;

$\Rightarrow$ f_0 is the <u>Fundamental Frequency</u>

$k \cdot f_0$ are <u>Harmonic</u> frequencies of the signal. (k integer)

Ex: $x(t) = \cos(2\pi(3)t) + \cos(2\pi(4.5)t)$

Is $x(t)$ periodic?

$f_1 = 3\,Hz \qquad (f_{-1}) = -3\,Hz$

$f_2 = 4.5\,Hz \qquad (f_{-2}) = -4.5\,Hz.$

$f_0 = 1.5\,Hz \qquad \left(\frac{3}{1.5}\right)$ integer 2 ✓ $\quad \left(\frac{4.5}{1.5}\right)$ ✓ integer 3 ✓

fundamental frequency

<u>Def:</u> The frequencies $(k \cdot f_0)$; that is integer multiples of f_0 are called the <u>harmonics</u> of f_0.

Fact: When we add sinusoids w/ frequencies that are harmonics of a fundamental frequency f_0, then we get a periodic signal.

Ex: $x(t) = \cos(2\pi(5.5)t) + 2\sin(2\pi(7.5)t)$

Is $x(t)$ periodic? Yes

$f_0 = 0.5\,\text{Hz}$ fund. freq.

$\frac{5.5}{0.5} = 11 \Leftarrow k$ $\qquad$ $\frac{7.5}{0.5} = 15 \Leftarrow k$

$\rightarrow$ We have 11^{th} & 15^{th} harmonics.

What are a_ks? spectrum.

$$x(t) = \sum a_k e^{j2\pi k f_0 t}$$

$$x(t) = \frac{1}{2} e^{j2\pi(5.5)t} + \frac{1}{2} e^{-j2\pi(5.5)t} + \frac{1}{j} e^{j2\pi(7.5)t} - \frac{1}{j} e^{-j2\pi(7.5)t}$$

$\frac{1}{j} = e^{-j\pi/2}$

a_k's:

$a_{11} = \frac{1}{2}$

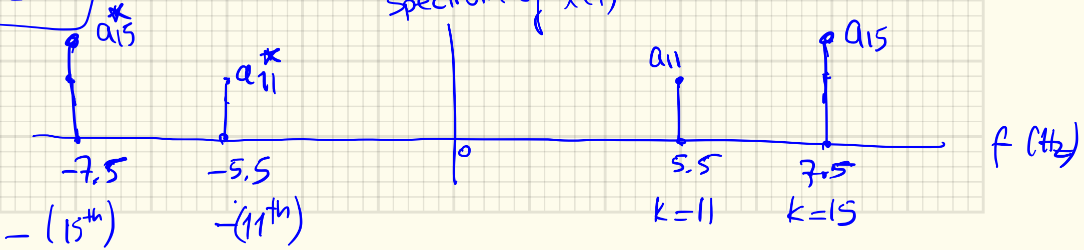
$a_{-11} = \frac{1}{2}$

$a_{15} = e^{-j\pi/2}$

$a_{-15} = e^{j\pi/2}$

other $a_k = 0$

Spectrum of $x(t)$



Exercise (HW): $x(t) = 4 + \cos^2(4\pi t) + \sin(32\pi t)$ $\quad f_0 = 2\,\text{Hz}$? X.

What is the fundamental frequency of this signal ?

— What harmonic frequencies exist ?

— Plot the spectrum.

Reading : Start chapter 3.

Next time Fourier Series.