BLG 354E Signals & Systems

Week 4

22.03.2021

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In general:

Signal Representation in terms of Basis Functions :

$$\boxed{\underset{\uparrow}{x[n]} = \sum_k \underset{\text{coefficients}}{\underset{\sim}{a_k}} \boxed{\psi_k[n]}}$$

$\downarrow$ Basis functions

↳ Important case : $\psi_k[n]$ are orthogonal bases

$$\sum \psi_k[n] \, \psi_\ell^*[n] = 0$$

$$\equiv \langle \psi_k , \psi_\ell \rangle = 0$$

special case. Discrete Fourier series: $e^{j\frac{2\pi}{N}kn}$.

$$\left\{ \psi_k[n] \right\}_{k=1}^{\infty ; N}$$: any set of basis functions/signals.

Desirable Goal → Small # of $\{a_k\}$ are non-zero. : (Compression)

Keep $\{a_k\}$ that contain the signal, : Filtering ←

discard $\{a_k\}$ that contain $\underbrace{\text{"noise"}}_{\text{any unwanted features}}$

General Theory of Fourier Series

Any periodic signal $x(t)$ w/ fundamental frequency f_0, can be written as a sum of harmonically related sinusoids.

$k \cdot f_0$: harmonic frequencies (k integer)

Fourier Synthesis Eqn:

$$x(t) = \sum_{k=-\infty}^{\infty} \boxed{a_k} \, \underbrace{e^{j2\pi k f_0 t}}_{\psi_k(t)}$$

a_k : Fourier Series coefficients

$\psi_k(t)$: basis functions.

↳ building blocks for reconstructing $x(t)$.

Note that the basis fn:

$$\psi_k(t) = \boxed{e^{j2\pi k \cdot f_0 t}} \longrightarrow \text{Complex exponential (sinusoidals)}$$

index of the basis fn.

fundamental freq.

↳ basis functions for Fourier series.

Synthesis means:

→ Given a_k (for a certain signal $x(t)$) & f_0 (fundamental freq); $\boxed{\frac{1}{f_0} = T_0}$; Fundamental Period of $x(t)$

I can reconstruct the signal $x(t)$ as follows:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \underbrace{e^{j2\pi k . \underline{f_0} t}}$$

more generally:

$$\left(x(t) = \sum_{k} a_k \psi_k(t) \right)$$

Family of **Basis functions**: $\left\{ V_{\underline{k}}(t) = e^{j2\pi \underline{k} f_0 t} \right\}_{k=-\infty}^{\infty}$ → has 2 properties

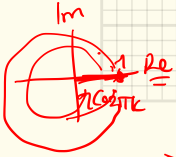
Property 1 Zero-Integral ::

Integrate the basis functions in 1 period.

$$\int_0^{T_0} V_k(t)\,dt = \int_0^{T_0} e^{j2\pi k f_0 t}\,dt = \left. \frac{e^{j2\pi k f_0 t}}{j2\pi k f_0} \right|_0^{T_0}$$

$$= \frac{1}{j2\pi k f_0} \left(\underbrace{e^{j2\pi k . \overbrace{f_0 . T_0}^{1}}} - 1 \right) = 0$$

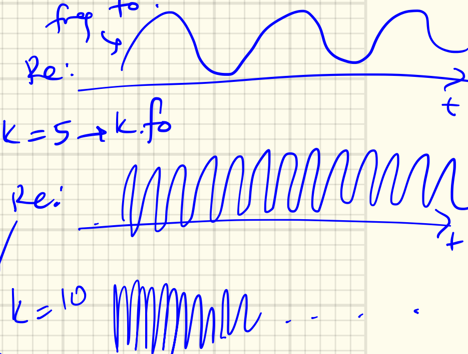
$$\left(e^{j2\pi k} - 1 \right)$$



Re $= \cos 2\pi k + j \underline{\sin 2\pi k}$

$\underbrace{e^{j2\pi k}} = \underline{1}$ for any $\underline{\text{integer}}$ k

$k = \underline{1}$ $\quad e^{j2\pi f_0 t}$ → cos, → sin



for $k=0$: $\int_0^{T_0} \underbrace{e^{j2\pi (0) f_0 t}}_{1}\,dt = T_0$ ⟹

Property 1)

$$\int_0^{T_0} v_k(t)\,dt = \int_0^{T_0} e^{j2\pi k f_0 t}\,dt = \begin{cases} 0 & , \quad k \neq 0 \\ T_0 & , \quad k = 0 \end{cases}$$

Property 2: Orthogonality of harmonic complex exponentials

$$\int_0^{T_0} v_k(t) \cdot v_\ell^*(t)\,dt = 0 \qquad \text{for } k \neq \ell \quad , \quad k,\ell : \text{integers}$$

Let's show this:

$$\int_0^{T_0} v_k(t) \cdot v_\ell^*(t)\,dt = \int_0^{T_0} \underbrace{e^{j2\pi k f_0 t} \cdot \underbrace{e^{-j2\pi \ell f_0 t}}_{}}_{}\,dt$$

$$= \int_0^{T} e^{j2\pi \underbrace{(k-\ell)}_{\substack{m=k-\ell \\ \text{integer}}} f_0 t}\,dt = \int_0^{T_0} e^{j2\pi m f_0 t}\,dt = \begin{cases} 0 & , \quad m \neq 0 \\ T_0 & , \quad m = 0 \end{cases}$$

$$\boxed{\int_0^{T_0} v_k(t)\, v_\ell^*(t)\,dt \overset{0}{=} \overset{k\neq\ell,}{} = \begin{cases} 0 & , \quad k \neq \ell \\ T_0 & , \quad k = \ell \end{cases}}$$

$$\boxed{\langle \underline{v_k(t)}, \underline{v_\ell(t)} \rangle = 0 \quad k \neq \ell}$$

✓ **Fourier Synthesis:** Given a_k & f_0 for $x(t)$: $\underbrace{x(t)}_{\text{we construct (synthesize } x(t)} = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$

Q: Given a signal $x(t)$, how to find a_k? $\rightarrow$ Fourier Series coefficients

(Fourier analysis)

A: Start with $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$

multiply both sides by

$$x(t) e^{-j2\pi l f_0 t} = \sum_k a_k e^{j2\pi k f_0 t} \cdot e^{-j2\pi l f_0 t}$$

integrate

$$\int_0^{T_0} x(t) e^{-j2\pi l f_0 t} dt = \int_0^{T_0} \sum_k a_k e^{j2\pi k f_0 t} \cdot e^{-j2\pi l f_0 t} dt$$

$$= \sum_k a_k \underbrace{\int_0^{T_0} e^{j2\pi (k-l) f_0 t} dt}$$

$$\hookrightarrow = \begin{cases} 0, & k \neq l \\ T_0, & k = l \end{cases}$$

$$\int_0^{T_0} x(t) e^{-j2\pi l f_0 t} dt = \boxed{a_l \cdot T_0} \Leftarrow \sum_k a_k \cdot \underset{k \neq l}{0}$$

only surviving term

(change the index to k) $\rightarrow$

$$a_k = \frac{1}{T_0}\underbrace{\int_0^{T_0} x(t) . e^{-j2\pi k.f_0 t} \, dt}_{\langle x(t) , v_k(t) \rangle}$$

: projection of x(t) signal onto bases functions $\{v_k(t)\}_{k=-\infty}^{\infty}$

Fourier Analysis Integral

(how to calculate Fourier series coefficients) $\rightarrow$

$$\boxed{a_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t) e^{-j 2\pi k.f_0 t} \, dt .}$$

$\uparrow$

(integral limits can be any 1 period interval)

$$\frac{A}{2}\cos(2\pi f - \Delta) + \frac{A}{2}\cos(2\pi(f \pm \Delta)t$$

$$= A\left(\cos 2\pi f t\right)\left(\cos 2\pi\Delta t\right)$$

Check previous lecture.

Sanity check ⟷

Ex: Consider the beat note

$$x(t) = \cos(2\pi\, 100t) \cdot \cos(2\pi\, 10\, t)$$

$$\longrightarrow x(t) = \frac{1}{2}\cos(2\pi \boxed{110}\, t) + \frac{1}{2}\cos(2\pi \boxed{90}\, t)$$

Q: What are Fourier series coefficients $x(t)$? a_k

1. Q: What is the fundamental frequency of $x(t)$?

x(t) has Two frequencies → $\left\{ \begin{array}{l} f_1 = 110 \text{ Hz} \\ f_2 = 90 \text{ Hz} \end{array} \right)$ gcd. f_1, f_2 $\longrightarrow$ $\boxed{f_0 = 10 \text{ Hz}}$. fundamental frequency

$\rightarrow$ 9th & 11th harmonics exist in $x(t)$:

$a_9 = ?$, $a_{-9} = ?$

$a_{11} = ?$, $a_{-11} = ?$

$F\ 9\ f_0$, $F\ 11 \cdot f_0$

$F\ 90$ Hz $\quad F\ 110$ Hz

Use inverse **Euler formula**

Rewrite $x(t) = \frac{1}{4}\boxed{e^{j2\pi(11)(10)t}}_{k \cdot f_0} + \frac{1}{4}e^{j2\pi(-11)\cdot 10 \cdot t}$

$$+ \frac{1}{4}e^{j2\pi(90)t} + \frac{1}{4}e^{j2\pi(-90)t}$$

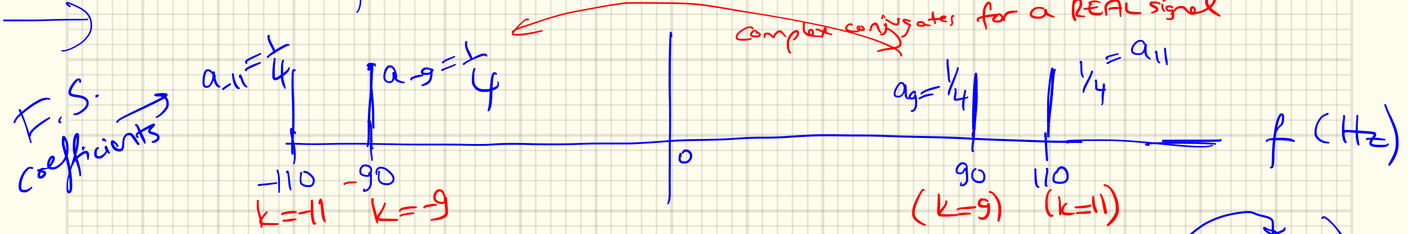
$9 \cdot 10 \quad -9 \cdot 10$

$a_9 = \frac{1}{4} \quad a_{11} = \frac{1}{4}$

$a_{-9} = \frac{1}{4} \quad a_{-11} = \frac{1}{4}$

Spectrum representation of x(t) (Fourier series representation in graphical form)

→



ex: if

$$x(t) = 3\,\underline{\sin(2\pi 5 t)} = \frac{3}{2j}\left(e^{j2\pi 5t} - e^{-j2\pi 5t}\right)$$

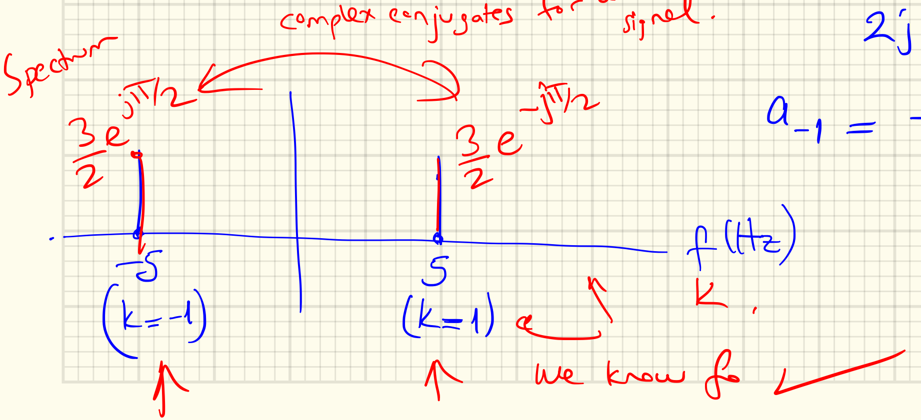
$k f_0$ $1 \cdot f_0$ $(-1) \cdot f_0$

$f_0 = 5$ Hz $k = \mp 1$.

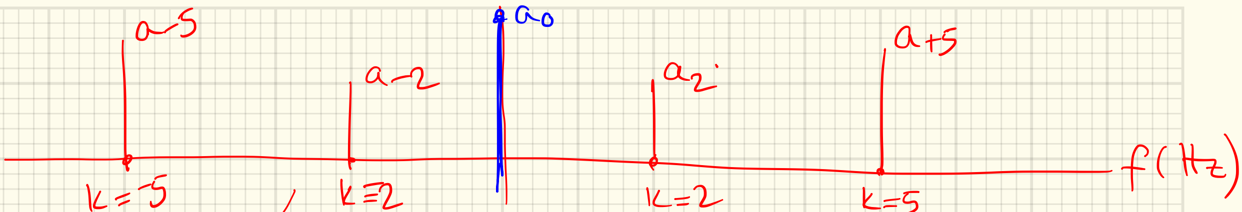
Q: F.S coeff of x(t)? a_k?

$$a_1 = \frac{3}{2j} = \boxed{\frac{3}{2} e^{-j\pi/2}}$$ → phasor form

$$a_{-1} = -\frac{3}{2j} = \boxed{\frac{3}{2} e^{j\pi/2}}$$



Given spectrum



Write down the signal

$$x(t) = a_0 + 2a_2\cos(2\pi 2 f_0 t) + 2a_5\cos(2\pi 5 f_0 t)$$

$$x(t) = a_0 + 2a_2\sin\left(2\pi f_0 t - \frac{\pi}{2}\right) + 2a_5\sin\left(2\pi 5 f_0 t - \frac{\pi}{2}\right)$$

F.S. analysis equations :
$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)\cdot e^{-j2\pi k f_0 t}\, dt$$

$k=0$

$$\boxed{a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\, dt}$$

integral (area under the signal $x(t)$ in 1 period, scaled by T_0.

* For a $\boxed{\text{real}}$ signal (not complex) & periodic:

a_k's have a special property: Let's derive it:

$$\left(x(t)\right)^* = \left(\sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}\right)^*$$

← Take complex conjugate on both sides

b/c $x(t)$ is real

$$= x(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-j2\pi k f_0 t}$$

Let's do a change of variables $k' = -k$

$$x(t) = \sum_{k'} \underbrace{a^*_{-k'}}\, e^{j2\pi k' f_0 t} = \underbrace{\sum_{k'} \underbrace{a_{k'}}\, e^{j2\pi k' f_0 t}}_{\text{F.S. representation}}$$

$$\Rightarrow a_k = a^*_{-k}$$: for a real periodic signal

$$\equiv \boxed{a_{-k} = a_k^*}$$

✓

→ Using this result ($a_{-k} = a_k^*$), we will derive an equivalent F.S. representation (synthesis eqn) using **real** sinusoid bases $\{\cos 2\pi k f_0 t\}$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

use Euler formula

$$x(t) = \cdots + \boxed{\overset{a_N^*}{a_{-N}}} e^{-j2\pi N f_0 t} + \cdots + \boxed{\overset{a_2^*}{a_{-2}}} e^{-j2\pi 2 f_0 t} + \boxed{\overset{a_1^*}{a_{-1}}} e^{-j2\pi f_0 t}$$

$$+ a_0 + \boxed{a_N} e^{j2\pi N f_0 t} + \cdots + \boxed{a_2} e^{j2\pi f_0 t} + \boxed{a_1} e^{j2\pi f_0 t} + \cdots$$

∞ly many

for real $x(t)$

$$x(t) = a_0 + 2\left[a_1 \cos(2\pi f_0 t) + a_2 \cos(2\pi(2f_0)t) + \cdots + a_N \cos(2\pi(Nf_0)t) + \cdots\right)$$

$$\Longrightarrow \boxed{x(t) = a_0 + 2\sum_{k=1}^{\infty} a_k \cos(2\pi(kf_0)t)}$$

Fourier series representation for a real signal (equivalent to the previous one but uses a Real Basis)

If we have phase:

$$x(t) = a_0 + \sum_k A_k \cos(2\pi k f_0 t + \phi_k)$$

→ F.S coefficients

$$a_k = \frac{A_k}{2} e^{j\phi_k} \Longleftrightarrow a_{-k} = \frac{A_k}{2} e^{-j\phi_k}$$

→ real signal !

Ex: $x(t) = \sin^3(3\pi t)$. Q. Find the Fourier series representation

" $\equiv$ " (coefficients)

You have to find fundamental freq.

A. Find f_0 : ? fundamental freq.

→ Show the spectrum of $x(t)$.

b/c w/ f_0 → $\{e^{j2\pi k f_0 t}\}_k$ we can express the harmonic complex exponential bases $(k.f_0)$.

A: Euler formula

$$x(t) = \left(\frac{1}{2j}\left(e^{j3\pi t} - e^{-j3\pi t}\right)\right)^3$$

$$= \frac{1}{-8j}\left(e^{j9\pi t} - 3e^{j6\pi t}\cdot e^{-j3\pi t} + 3e^{j3\pi t}\cdot e^{-j6\pi t} - e^{-j9\pi t}\right)$$

(Recall: $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$)

$$= \frac{e^{j\pi/2}}{8}\left(e^{j9\pi t} - 3\underline{e^{j3\pi t}} + 3e^{-j3\pi t} - e^{-j9\pi t}\right)$$

$\omega_0 = 3\pi$ rad/s

$f_0 = 1.5$ Hz.

1st & 3rd harmonics exist

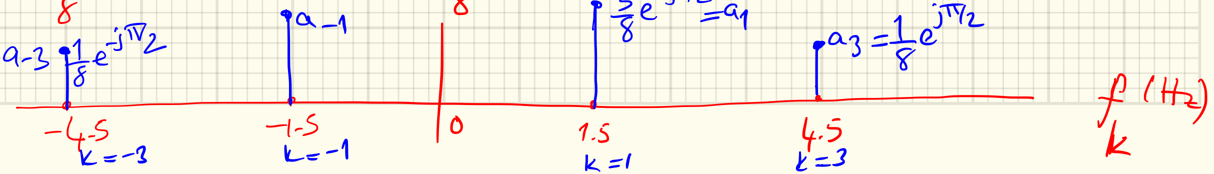
$a_{\mp 1}$, $a_{\mp 3}$

real signal

$$a_1 = -\frac{3}{8}e^{j\pi/2} = e^{-j\pi}e^{j\pi/2}\cdot\frac{3}{8} = \frac{3}{8}e^{-j\pi/2} \Rightarrow a_{-1} = \frac{3}{8}e^{j\pi/2}$$

$$a_3 = \frac{1}{8}e^{j\pi/2} \rightarrow a_{-3} = \frac{1}{8}e^{-j\pi/2}$$

Spectrum of $x(t)$



F.S. coef of $x(t)$ →

$$a_k = \begin{cases} 0, & k \neq 1, -1, 3, -3 \\ \frac{3}{8} e^{j\left(\mp\frac{\pi}{2}\right)}, & k = \pm 1 \\ \frac{1}{8} e^{j\left(\mp\frac{\pi}{2}\right)} & k = \mp 3 \end{cases}$$

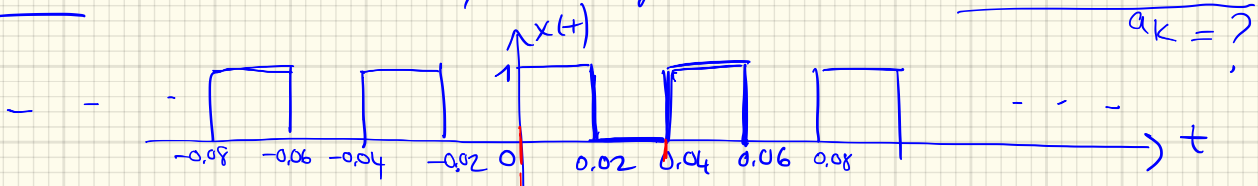
This was an example signal, already in sine form

Next, let's just work on a signal w/ a given waveform

→

Ex: Given $x(t)$ is the periodic square waveform. Calculate F.S. coefficients of $x(t)$.

$a_k = ?$



$T_0 = 0{,}04\,s$

$T_0 = ?$ 0,04 sec.

$f_0 = \frac{1}{T_0} = 25$ Hz ✓

project my signal onto this basis fn. set. $\longrightarrow$ $\left\{ e^{j2\pi k.25t} \right\}_k$ basis functions

$$a_k = \frac{1}{T_0} \int_0^{T_0} \underline{x(t)}\, e^{-j2\pi k f_0 t}\, dt \; ;$$

$$\boxed{a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt = \frac{1}{0{,}04}\,0.02 = \frac{1}{2}}$$

in 1 period of the signal

$$x(t) = \begin{cases} 1, & 0 \le t \le 0{,}02 \\ 0, & 0{,}02 < t \le 0{,}04 \end{cases}$$

express x(t) mathematically.

$$a_k = \frac{1}{0{,}04}\int_0^{0{,}04} x(t)\, e^{-j2\pi k 25 t}\, dt = \frac{1}{0{,}04}\left\{ \int_0^{0{,}02} 1 . e^{-j2\pi k 25t}\,dt + \cancel{\int_{0.02}^{0.04} 0 . e^{-j2\pi k 25t}\,dt} \right\}$$

$\xrightarrow{} 0$

$$a_k = 25 \left. \frac{e^{-j2\pi k 25 t}}{-j2\pi k 25} \right|_0^{0{,}02} = \cancel{25}\frac{\left(e^{-j2\pi k 25 (0.02)} - \overset{1}{\cancel{e^{-j0}}}\right)}{-j\,2\pi k \cancel{25}} = \frac{\left(1 - e^{-j\pi k}\right)}{j\,2\pi k} \longrightarrow$$

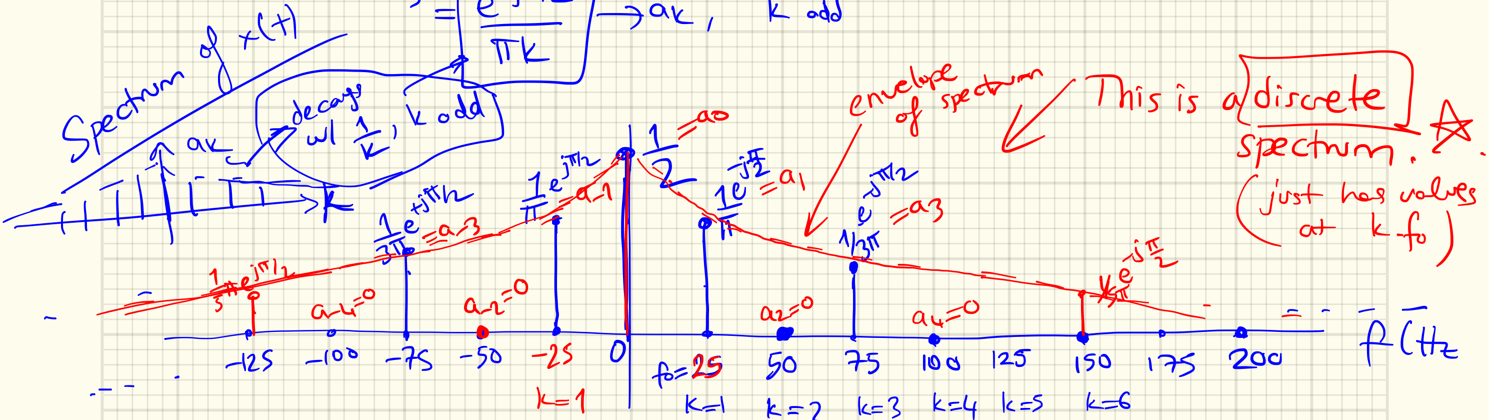
Notice: $e^{-j\pi k} = (e^{-j\pi})^k = (-1)^k = \begin{cases} 1, & k \text{ even} \\ -1, & k \text{ odd} \end{cases}$

$$\Rightarrow a_k = \begin{cases} \frac{1}{2}, & k=0 \\ 0, & k \text{ even} \\ \frac{1}{j\pi k}, & k \text{ odd} \end{cases}$$

$$\left(\frac{1}{j\pi k}\right) = \boxed{\frac{e^{-j\pi/2}}{\pi k}} \rightarrow a_k, \quad k \text{ odd}$$

Final eqn we got

$$a_k = \frac{1}{j2\pi k}\underbrace{\left(1-(-1)^k\right)}_{\begin{cases} 2 & \text{for } k \text{ odd} \\ 0, & \text{for } k \text{ even} \end{cases}}$$



This is a discrete spectrum. (just has values at $k \cdot f_0$)

$$\boxed{x_N(t) \triangleq \sum_{k=-N}^{N} a_k e^{j2\pi k 25 t}}$$

Goal ⟶

Error we make w/ the approximation $\underset{\sum}{\underline{x_N(t)}}$

$$e_N(t) = \overset{\max}{\left| x(t) - x_N(t) \right|} \xrightarrow[N\to\infty]{} 0$$

(worst-case error)

→ 0 as $N \to \infty$

for a continuous signal

But for a discontinuous signals: ~~$\max e_N(t) \xrightarrow[N\to\infty]{} 0$~~ → Gibbs phenomena.

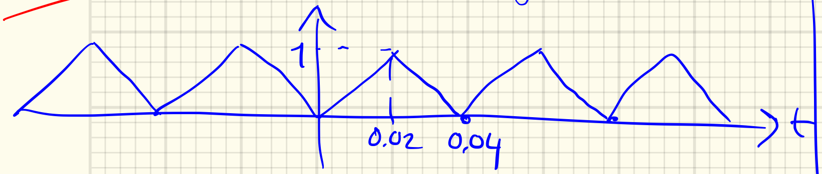
↓

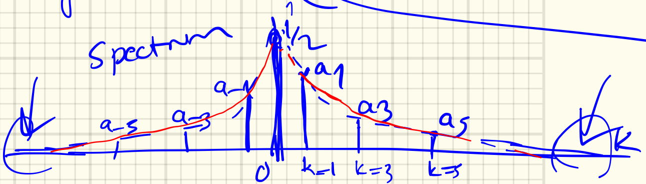
Visible at discontinuities of the signal

For a $\boxed{\text{continuous signal}}$ $\max e_N(t) \xrightarrow[N\to\infty]{} 0$.

eg. check the square waveform!

Ex! for a triangular periodic signal /waveform





$T_0 = 0.04$, $f_0 = 25$ Hz.

F.S. coefficients: $a_0 = \frac{1}{0.04} \underline{0.02} = \frac{1}{2}$

$\boxed{a_k \propto \frac{1}{k^2}}$ for odd k. → decays faster than the a_k's of square wave.

not harmonically related.

Ex: $x(t) = \sin(2\pi \boxed{\sqrt{2}}\, t) + 2\sin(2\pi \boxed{4}\, t)$

Q: Can you find a Fourier Series representation of this signal?

$f_1 = \sqrt{2}$ Hz

$f_2 = 4$ Hz.

$\rangle$ No gcd of f_1 & f_2 ∃ no f_0 !

⇒ This is not a periodic signal ∴ No F.S. representation,

$x(t)$, Non-periodic signal

⇒ Study the Notes + Slides