BLG 354E Signals & Systems

Week 5

29.03.2021

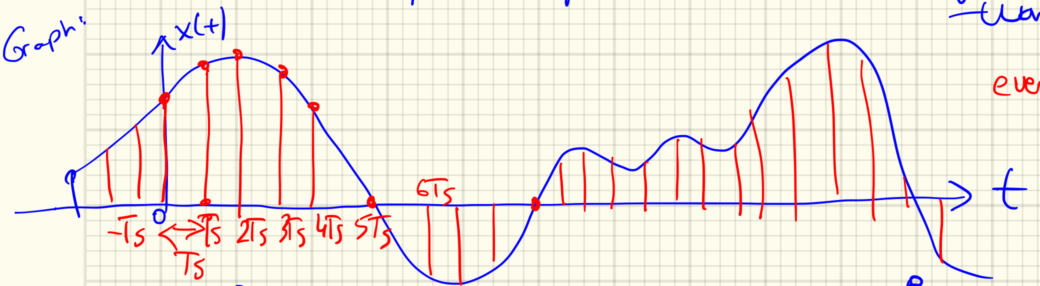
Gözde ÜNAL

Discrete-time Signals (DT)

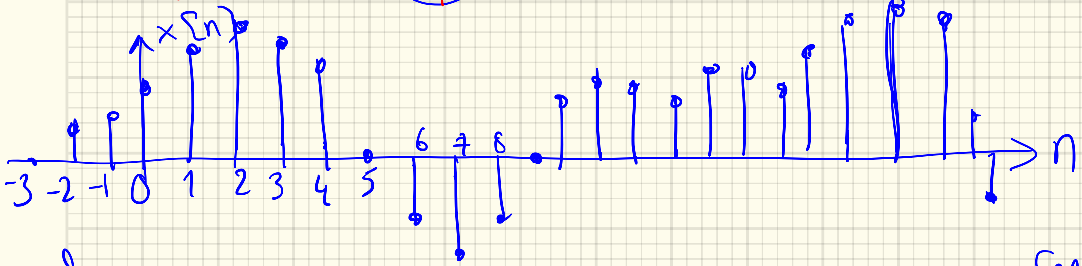
1D signal

1D. $x[n]$: a sequence of numbers indexed by integers, an integer variable n.

Graph:



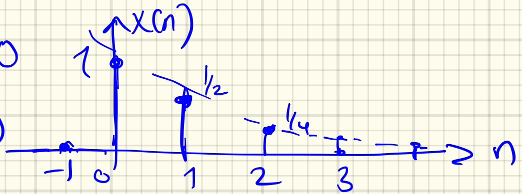
~~want to obtain~~ A sampled signal:
every T_s samples → sampling period



Sampled signal:

Functional

$$x[n] = \begin{cases} \left(\frac{1}{2}\right)^n & , n \geq 0 \\ 0 & , n < 0 \end{cases}$$



Sequence / array

$x[n] = [\, \cdots \; 0 \;\; 1 \;\; \frac{1}{2} \; \cdots]$

↑
n=0

Tabular:

n	…	-2	-1	0	1	2	3	4
x(n)	…	0	0	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	…

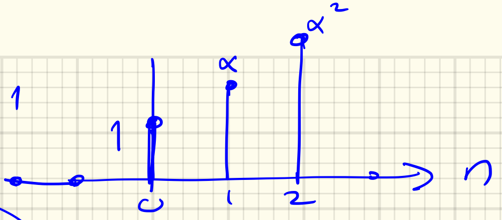
Elementary DT Signals:

* $x[n] = A\alpha^n$:

$\downarrow$

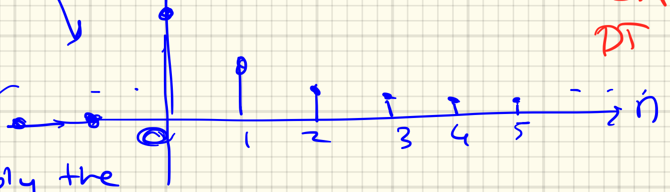
$0<\alpha<1$

$\alpha > 1$



CT : $x(t) = A(\underbrace{e^{-a}}_{\alpha})^t = Ae^{-at}$

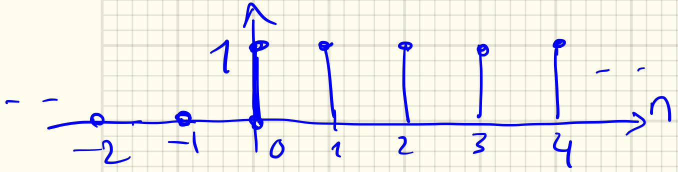
$a>0$



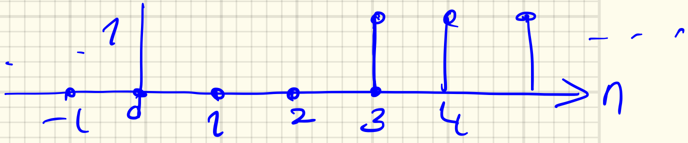
Exponential DT signal

We need a step fn. multiply the signal to make it zero on the negative.

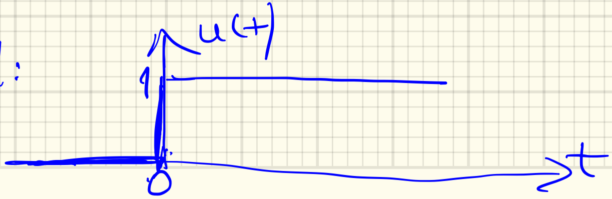
* $x[n] = u[n]$: Unit Step Sequence



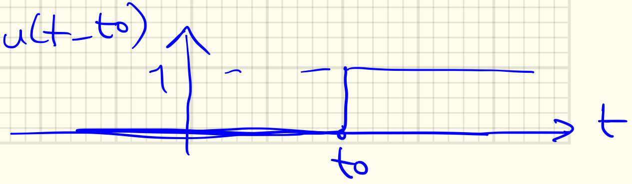
$u[n-3]$



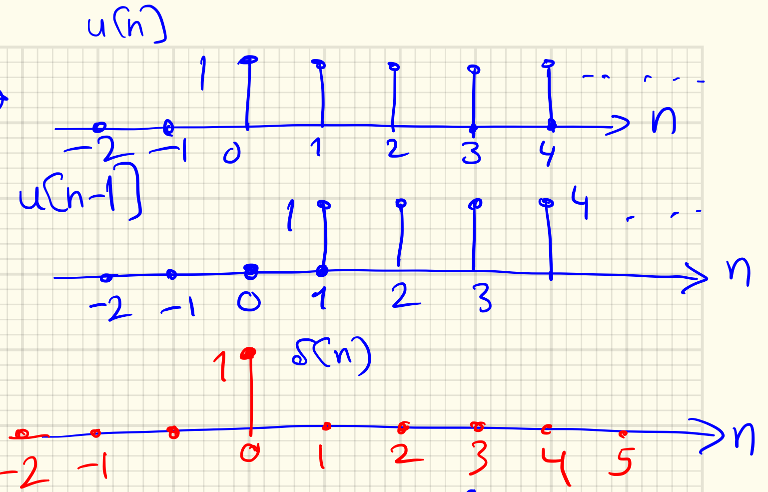
Recall:



$u(t-t_0)$



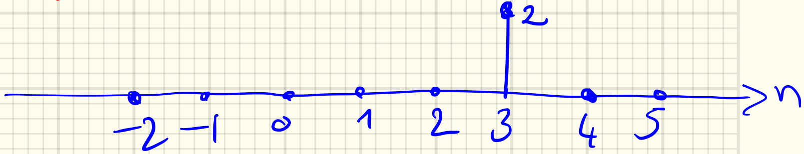
$\rightarrow u[n] - u[n-1]$



$\triangleq \delta[n]$

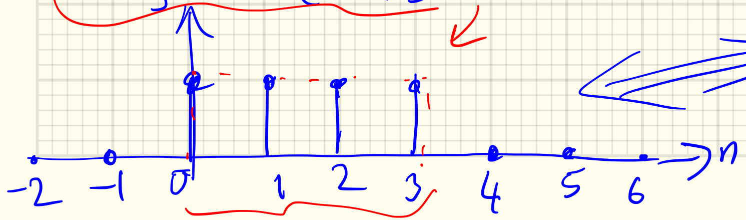
unit impulse sequence

$2\,\delta[n-3] \longrightarrow$



$$\delta[n] = \begin{cases} 1 & , n=0 \\ 0 & , n \neq 0 . \end{cases}$$

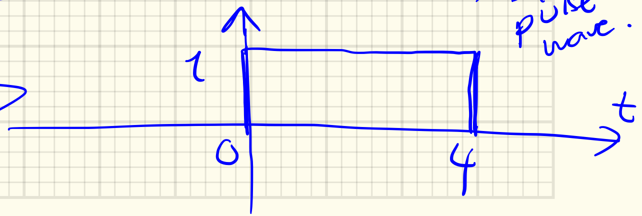
$u[n] - u[n-4]$:



$\longrightarrow$ CT version

$u(t) - u(t-4)$

square pulse wave.



Sinusoidal DT Sequences :

$t \rightarrow n$

$$\boxed{t = n.T_s}$$

Recall CT sinusoid : $x(t) = A \cos(\omega t + \phi)$

uniform sampling system

$$x(t) \longrightarrow \boxed{\text{A/D Converter}} \longrightarrow x(nT_s) = x(t)\Big|_{\boxed{t = nT_s}}$$

$x[n]$ ↑ output

$\rightarrow x[n]$: DT signal

parameter of the system

$= C/D$ w/ T_s sampling period

$f_s = \frac{1}{T_s}$: sampling frequency

DT Sinusoid:

$x(nT_s) \Rightarrow x[n]$

$$= A \cos(\omega \overbrace{T_s\, n}^{t} + \phi) \Longrightarrow \boxed{A \cos(\hat{\omega}\, n + \phi)} = x[n]$$

$$\boxed{\hat{\omega} \triangleq \omega T_s} = \frac{\omega}{f_s} \Big\} \text{ normalized frequency } \hat{\omega}.$$

Sampling Theorem : We should pick a sampling frequency (to sample a CT signal) at least $2 \cdot f_0$

(we'll study later in Fourier domain)

say ($\sin \omega_0 t$)

$$\boxed{f_s > 2 \cdot f_0}$$

DT Sinusoid:

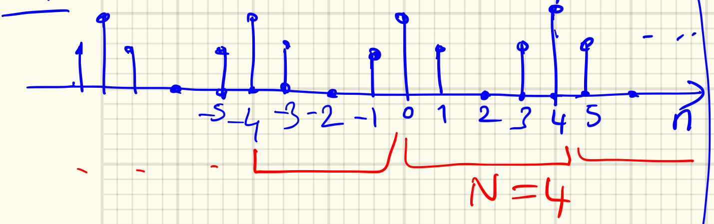
$x[n] = \cos(\hat{\omega}_0 n) = \cos((\hat{\omega}_0 + 2\pi k) n)$

To have a periodic DT sinusoid $=$

↓

fundamental period $N \Rightarrow x(n) = x[n+N] = \cos(\hat{\omega}_0 (n+N))$

ex: Fundamental Period $N=4$



$\hat{\omega}_0 . N = 2\pi \underline{k}$

$\left(\frac{\hat{\omega}_0}{2\pi}\right) = \left(\frac{k}{N}\right) \rightarrow$ if this is a rational number

$\Rightarrow$ I have a DT sinusoid.

$\hat{\omega}_0 = \omega_0 . T_s$

$2\pi \hat{f}_0 = 2\pi f_0 . T_s$

$\boxed{\hat{f}_0 = f_0 . T_s} = \frac{f_0}{f_s}$: normalized freq. by sampling freq.

$\hat{f}_0 = \frac{k}{N}$

$f_0 . T_s = \frac{k}{N}$

$\boxed{k . T_0 = N . T_s}$ can write it also as $\leftarrow$ $\boxed{\frac{f_0}{f_s} = \frac{k}{N}} \rightarrow \boxed{f_0 . N = k . f_s}$

Fact about

⇒ DT sinusoids: If you shift the frequency by integer multiples of 2π you end up w/ the same sinusoid.

$$\hat{\omega}_2 = \hat{\omega}_1 + m.2\pi \rightarrow \sin(\hat{\omega}_2 n) = \sin((\hat{\omega}_1 + m2\pi)n)$$

easy to show (by trig identity)

$$\sin((\hat{\omega}_1 + m2\pi)n) = \sin(\hat{\omega}_1 n).\underbrace{\cos(mn2\pi)}_{1} + \cos(\hat{\omega}_1 n).\underbrace{\sin(mn2\pi)}_{0}$$

✓

Note 1 DT sinusoids are only unique over

$$\boxed{0 \le \hat{\omega} < 2\pi}$$

could be any 2π interval

(Contrast this to CT freq. ω. $\hat{\omega} \Longleftrightarrow \omega$ — does not have this prop.)

Note 2: Rate of oscillations of a DT sinusoid increases from $\hat{\omega}_0 : 0 \nearrow \pi$, then decreases as $\hat{\omega}_0$ goes $\pi \longrightarrow 2\pi$.

★ Low frequencies of DT sinusoids are in vicinity of $\hat{\omega}_0 = 0 \mp 2k\pi$, k integer.

zero freq.

High frequencies " $\hat{\omega}_0 = \pi \mp 2k\pi$

π rad. freq.

Fourier Series Representation for DT Signals:

Synthesis eqn for a periodic signal x[n] w/ period N N: (integer)

$$x[n] = \sum_{k=0}^{N-1} \underbrace{a_k}_{} e^{j\frac{2\pi}{N}kn}$$

a_k → F.S. coefficients of a DT periodic signal

$$\boxed{\Phi^k[n] = \left(e^{j\frac{2\pi}{N}k}\right)^n}$$

Fourier basis fn

↳ really complex DT sinusoids

Recall: F.S. for CT signals

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \boxed{e^{jk2\pi f_0 t}}$$

∞ sum.

Basis fn. in the CT case.

period N

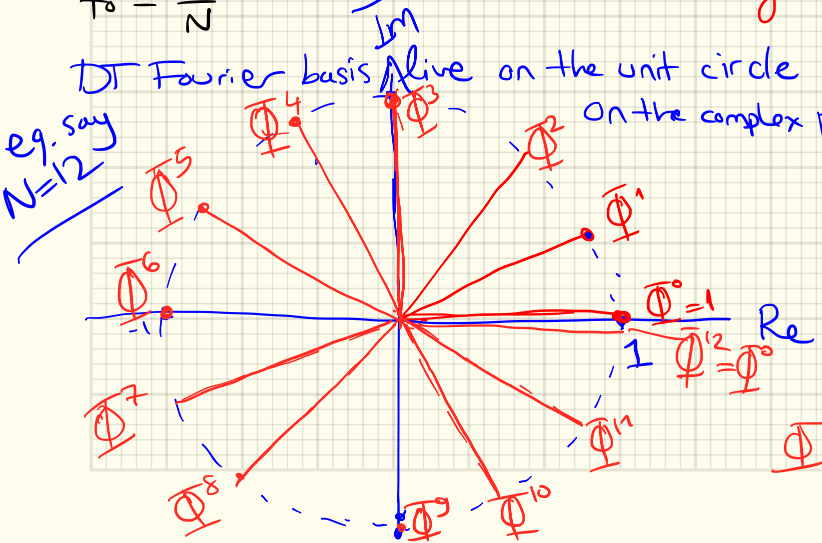
$\hat{\omega}_0 = \frac{2\pi}{N}$

$\hat{f}_0 = \frac{1}{N}$

Note: Complex sinusoidal DT.

$x(n] = Ae^{j\hat{\omega}_0 n} = A\cos(\hat{\omega}_0 n) + jA\sin(\hat{\omega}_0 n)$

DT Fourier basis live on the unit circle on the complex plane



$\Phi^0 = 1$

$\Phi^1 = e^{j2\pi/12}$

$\Phi^2 = e^{j2\pi/3}$

$\Phi^3 = e^{j\pi/2}$

$\Phi^6 = e^{j2\pi/12 \cdot 6} = e^{j\pi}$

Sum of the basis

$$\sum_{n=0}^{N-1} \underbrace{e^{j\frac{2\pi}{N}kn}}_{\Phi_k(n)} = N$$

for k=0, ∓N, ∓2N, ..

$$\sum_{n=0}^{N-1} \Phi^k(n) = 0$$

k ≠ 0, ∓N, ∓2N

How to obtain

→ Analysis eqn?

Start from synthesis

$$\left[x[n] = \left(\sum_{k=0}^{N-1} a_k e^{j\frac{2\pi}{N}\boxed{k}n} \right) \right] \cdot e^{-j\frac{2\pi}{N}\boxed{m}n}$$

$$\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}mn} = \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} a_k e^{j\frac{2\pi}{N}(k-m)n}$$

$$\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}mn} = \underbrace{\sum_{k=0}^{N-1} a_k}_{k=m.} \quad \underbrace{\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-m)n}}_{=0 \text{ for } k\neq m}$$

zero-sum property.

for $k=m$ $\quad \sum_{n=0}^{N-1} \underbrace{e^{j0}}_{1} = N$

Analysis eqn.

$$\Rightarrow \boxed{a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}}$$

how to calculate a_k (F.S. coef) given $x[n]$ & N ✓

⇒ For D.T. signals → their DTFS ; (Discrete-Time Fourier Series)

$$a_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\; \underbrace{e^{-j\frac{2\pi}{N}(k+N)n}}_{e^{-j\frac{2\pi}{N}kn} \cdot \underbrace{e^{-j\cancel{\frac{2\pi}{N}N}n}}_{=1}}$$

$$a_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

$$\boxed{a_{k+N} = a_k}$$

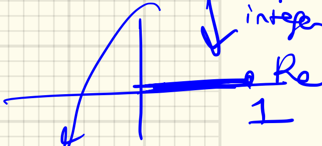
Fourier series coefficients for a DT periodic signal is also periodic w/ period N.

0.01.

$$\left(e^{j2\pi t} \neq 1\right)$$

but → $e^{j2\pi n}$

integer



always ends up on Real line at 1.

Summarize

DT. Fourier Series for a periodic x(n) w/ N (period)

Synthesis

$$x[n] = \sum_{k=0}^{N-1} a_k\, e^{j\frac{2\pi}{N}kn}$$

Analysis

$$a_k = \frac{1}{N}\sum_{k=0}^{N-1} x(n]\, e^{-j\frac{2\pi}{N}kn}$$

freq domain representation of x(n)

~~Extension~~

Late but just to note here: (You are using in your homeworks).

For a (non-periodic any) DT signal $x[n]$

$$x[n] = \sum_{k=-\infty}^{\infty} a_k \, e^{j\frac{2\pi k}{N} n}$$

n: time variable

k: frequency variable.

$$a_k = \frac{1}{N} X[k]$$

DFT: $\rightarrow \underbrace{X[k]}_{\text{freq domain representation of } x(n)} = \sum_{n=-\infty}^{\infty} x[n] \underbrace{e^{-j\frac{2\pi}{N} k.n}}$

in practice $\left(\sum_{n=-M}^{M} \right.$

$\boxed{N\text{-point DFT}}$

→ Fast Fourier Transform (fft): $\underset{\text{time-domain}}{x[n]} \rightarrow \underset{\text{freq-domain}}{X[k]}$

Inverse " " " ifft: $X[k] \longrightarrow x[n]$: time.

ex: (DSPFirst Slides 09 →) from Sampling thm

Ex: $x(t) = \cos(\overbrace{2\pi(100)}^{w_0 = 2\pi 100} t)$ → $w_0 = 200\pi$ rad/s

$f_0 = 100$ Hz

$T_0 = 0.01$ s.

$f_s > 2f_0$

$f_s > 200$ Hz.

$\boxed{T_s < 5 \text{ msec.}}$

① Sample x(t) at $\boxed{T_s = 0.5 \text{msec.}}$

$f_s = 2$ kHz.

$\hat{w}_0 = w T_s = 200\pi . 5 . 10^{-4} = 0.1\pi$ rad.

$= 2\pi(0.05)$ rad.

Sampled signal

→ $x[n] = \cos(0.1\pi n) = \cos(2\pi\ 0.05 n)$

→ $N = 20$: fundamental period

$\frac{\hat{w}_0}{2\pi} = \frac{k}{N}$

$\frac{2\pi . 0.05}{2\pi} = \frac{k}{N}$ ← 1

$N = \frac{1}{0.05} = 20$

$a_k = ?$ Fourier series coeff ?

$$\boxed{a_k = \frac{1}{20} \sum_{n=0}^{19} x[n] e^{-j\frac{2\pi}{20}kn}}$$

: F.S. Analysis eqn.

Recall $a_{k+20} = a_k$

Note: Do not use F.S. analysis eqn. here! I have already cosine exp. ⟹

$$x[n] = \cos(0.1\pi n) = \overset{a_1}{\boxed{\frac{1}{2}}} e^{j0.1\pi n} + \overset{a_{-1}}{\boxed{\frac{1}{2}}} e^{-j0.1\pi n}$$

$a_{-1} = a_{19}$

$a_{-1+20} = a_{19}$

from the synthesis formula!

$$x[n] = a_0 + \boxed{a_1} e^{j\frac{2\pi}{20}n} + a_2 e^{j\frac{4\pi n}{20}} + a_3 e^{j\frac{6\pi}{20}n} + \cdots + a_{18} e^{j\frac{2\pi}{20}18n} + \boxed{a_{19}} e^{j\frac{2\pi}{20}19n}$$

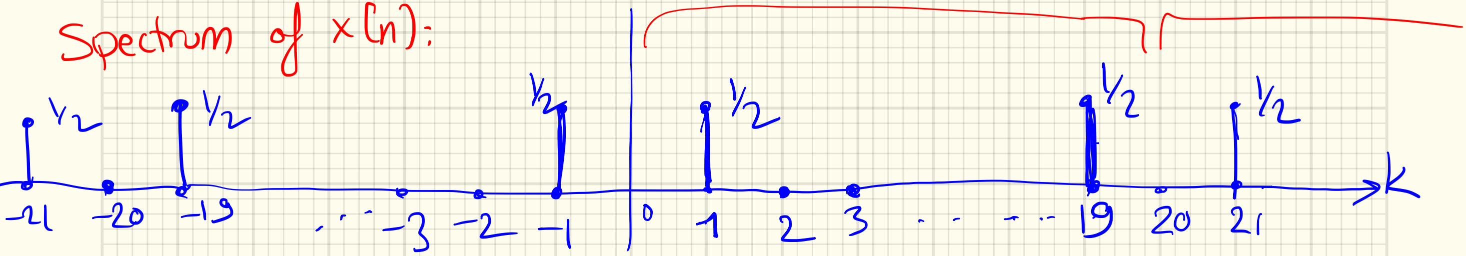
$$\Updownarrow$$

$$x[n] = \sum_{k=0}^{N-1=19} a_k e^{j\frac{2\pi}{20}kn}$$

$a_{k+20} = a_k$ $\qquad a_1 = \frac{1}{2}$, $\quad a_{19} = a_{-1} = \frac{1}{2}$

$a_0 = a_2 = a_3 = a_4 = \cdots = a_{18} = 0 \quad \cdots$

Spectrum of $x[n]$:



We just look at 1 period in freq domain too. ✓ $\qquad a_{k+20} = a_k$

② Now Sample the same CT signal at $T_s = 2ms$. ✓ w.r.t the sampling thm constraint.

$f_s = 500$ Hz.

We obtain DT sinusoid: $x[n] = \underbrace{\cos(2\pi\, 100 \cdot 2 \cdot 10^{-3}\, n)}_{\cos(0.4\pi n)}$

$\omega_0 = 2\pi . 100$

$\hat{\omega}_0 = \frac{\omega_0}{f_s} = 2\pi\, 100 . 2 . 10^{-3}$

$\left(\frac{\hat{\omega}_0}{2\pi}\right) = \left(\frac{0.4\pi}{2\pi}\right) = \frac{k}{N} \rightarrow 5$

Sampling period $\underline{N = 5}$ → defines the F.S. basis fn.

F.S. representation of $x(n)$

↳ $x[n] = \sum_{k=0}^{4 = N-1} a_k \boxed{e^{j\frac{2\pi}{5}kn}} \rightarrow a_{k+5} = a_k \rightarrow \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \rightarrow$ just calculate these.

Q: Find & plot the spectrum of $x[n]$ in this case.

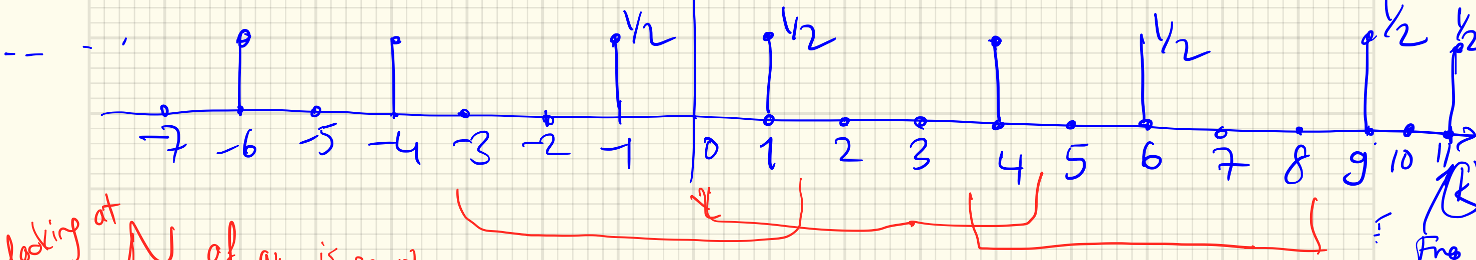
$$x[n] = \cos(0.4\pi n) = \frac{1}{2}e^{j\frac{2\pi}{5}n} + \frac{1}{2}e^{-j\frac{2\pi}{5}n}$$

$$x[n] = a_0 + a_1 e^{j\frac{2\pi}{5}n} + a_2 e^{j\frac{4\pi}{5}n} + a_3 e^{j\frac{6\pi}{5}n} + a_4 e^{j\frac{8\pi}{5}n} \quad ; \text{which } a_k \text{?}$$

$$a_1 = \frac{1}{2} \quad , \quad a_{-1} = \underbrace{a_4}_{a_{-1+5}\ \checkmark} = \frac{1}{2}$$

Spectrum of $x[n]$:

$\boxed{a_k \text{ are periodic}}$ $\quad a_{1+5} = a_6 = \frac{1}{2}$



looking at N of a_k is enough

any 5 sample period. ✓

exercise: $x[n] = \sin\left(\frac{6\pi}{N} n\right)$ $\sin\left(\frac{2\pi}{N} m.n\right)$

Calculate & plot spectrum of $x[n]$; a_k ?

$\left(\frac{\hat{\omega}_0}{2\pi}\right) = \frac{k}{N}$, $\underline{N=5}$, $k=3$

Q: $a_k = ?$ & plot spectrum:

$a_3 = \frac{1}{2j} = a_{-2} = a_8$

$a_{-3} = -\frac{1}{2j} = a_2 = a_7$.

Ex: $x[n] = 1 + \sin\left(\underbrace{\frac{2\pi}{N}}_{\hat{\omega}_0} n\right) + 3\cos\left(\underbrace{\frac{2\pi}{N}}_{\hat{\omega}_0} n\right) + \cos\left(\underbrace{\frac{4\pi}{N}}_{2\hat{\omega}_0} n + \frac{\pi}{2}\right)$

Calculate D.T.F.S coefficients for $x(n)$?

$N = ?$ N ✓

$$x[n] = \underbrace{1}_{a_0} + \underbrace{\frac{1}{2j}}_{a_1^{+}} e^{j\frac{2\pi}{N}n} \underbrace{- \frac{1}{2j}}_{a_{-1}} e^{-j\frac{2\pi}{N}n} + \underbrace{\frac{3}{2}}_{a_1^{+}} e^{j\frac{2\pi}{N}n} + \underbrace{\frac{3}{2}}_{a_{-1}} e^{-j\frac{2\pi}{N}n}$$

$$+ \underbrace{\frac{1}{2}}_{a_2} e^{j\pi/2} e^{j\frac{4\pi}{N}n} + \underbrace{\frac{1}{2}}_{a_{-2}} e^{-j\frac{\pi}{2}} e^{-j\frac{4\pi}{N}n}$$

$\hat{\omega}_0 = \frac{2\pi}{N}$

$a_0 = 1$

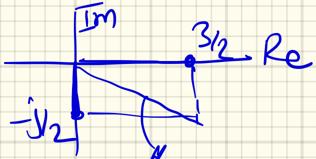
$a_1 = \frac{1}{2j} + \frac{3}{2} \rightarrow |a_1| = ?$

$\measuredangle a_1 = ?$

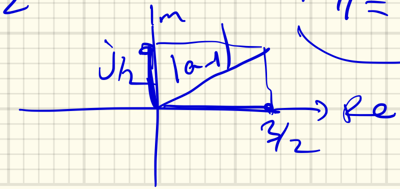
$a_{-1} = a_{N-1} = \frac{3}{2} - \frac{1}{2j} = \frac{3}{2} + \frac{j}{2}$

$a_2 = \frac{1}{2} e^{j\pi/2}$

$a_{-2} = a_{N-2} = \frac{1}{2} e^{-j\pi/2}$



$|a_1| = \frac{\sqrt{10}}{2}$ $\measuredangle a_1 = \text{atan}\left(\frac{-1}{3}\right)$



$a_1 = \frac{\sqrt{10}}{2} e^{j\,\text{atan}\left(\frac{-1}{3}\right)}$

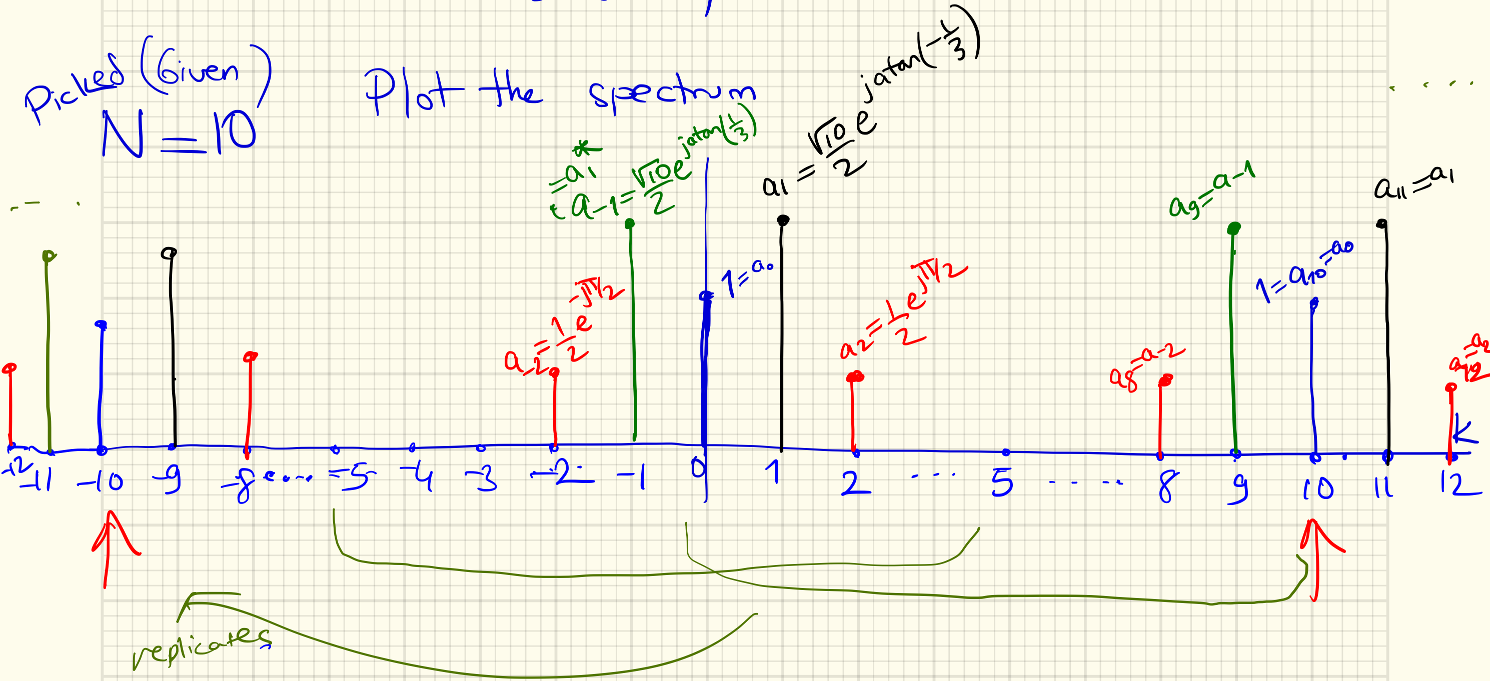
a_{-1}^{*}

Note : $\boxed{a_{-k} = a_k^*}$ Valid only for real signals

Recall (check your CTFS derivation)

Picked (Given) $N=10$

Plot the spectrum



a_k : $\underbrace{k=0,\dots,9}$ are unique , then replicates.