BLG 354E Signals & Systems

05.04.2021

Gözde ÜNAL

Recap: Any periodic DT signal:

Fourier synthesis

$$x[n] = \sum_{k=0}^{N-1} \underbrace{a_k}_{\text{coefficients}} \underbrace{e^{j\overbrace{\frac{2\pi}{N}}^{\hat{\omega}_0} k n}}_{\{\text{basis fns}\}_k}$$

Sequence $x[n]$ is periodic w/ fundamental period N.

You have to determine this

Fourier analysis

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \cdot e^{-j\frac{2\pi}{N} k n}$$

sum over time index

w/ a_k periodic w/ N.

; $a_{k+N} = a_k$

DTFS $\longleftrightarrow$ CTFS $\longrightarrow$ can have ∞ ly many distinct a_k s.

$$\sum_{k=-\infty}^{\infty} a_k e^{+j2\pi f_0 k t}$$

$\equiv$ harmonic component

finite $\Rightarrow N$

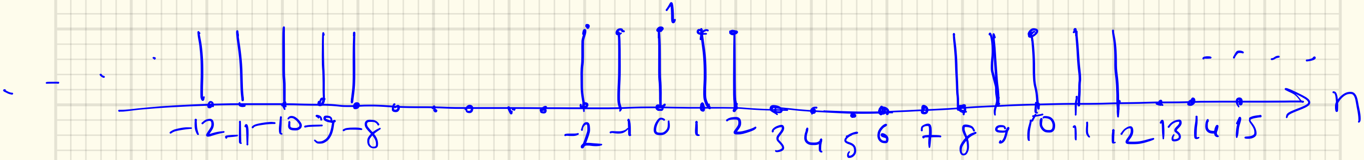
distinct harmonic components $e^{j\frac{2\pi}{N}kn}$.

Def: $|a_k|^2 \rightarrow$ its graph vs $\hat{f} = \frac{k}{N}$ or $\hat{\omega} = \frac{2\pi k}{N}$, or simply k, is known as the **power spectrum of** the periodic signal $x[n]$

Ex: Rectangular Pulse Train Sequence: say $N=10$;

In a period: $x[n] = \begin{cases} 1, & -2 \le n \le 2 \\ 0, & o/w \end{cases}$

$x[n+10]=x[n]$ ✓



Note CT version:

F.S. $a_k = \frac{1}{N}\sum_{n=-L=-2}^{L=2} 1\cdot e^{-j\frac{2\pi}{N}kn}$, $N = 10$

$k=0$: $a_0 = \frac{2L+1}{N} = \frac{5}{10} = \frac{1}{2}$ eg.

$$a_k = \frac{1}{10}\sum_{n=-2}^{2} 1\, e^{-j\frac{2\pi}{10}kn} \Rightarrow a_1 = \frac{1}{10}\sum_{n=-2}^{2} e^{-j\frac{2\pi}{10}n}$$

a_{-1}^{*}

$$a_1 = \frac{1}{10}\left(e^{+j\frac{4\pi}{10}} + e^{j\frac{2\pi}{10}} + 1 + e^{-j\frac{2\pi}{10}} + e^{-j\frac{4\pi}{10}}\right) = \frac{1}{10}\left(2\cos\left(\frac{4\pi}{10}\right) + 1 + 2\cos\left(\frac{2\pi}{10}\right)\right).$$

$= a_9 = a_{11} \ldots$

$a_{k+N} = a_k$

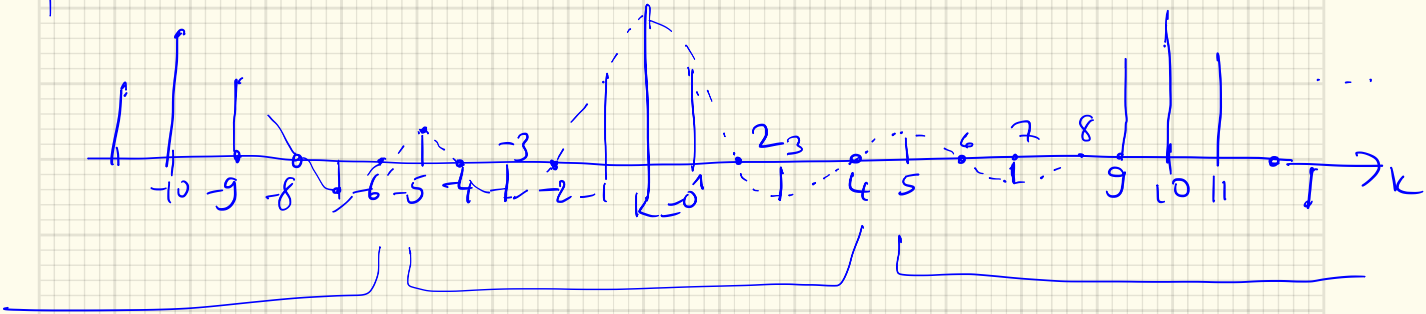
$$\Rightarrow a_k = \begin{cases} \dfrac{2L+1}{N}, & k = 0, \mp N, \mp 2N, \ldots \\ \dfrac{1}{N} \boxed{\dfrac{\sin\left(\frac{2\pi}{N} k \left(L + \frac{1}{2}\right)\right)}{\sin\left(\frac{2\pi}{N} k \cdot \frac{1}{2}\right)}}, & \text{o/w} \end{cases}$$

exercise derive this expression

for real signals $a_k = (a_{-k})^*$

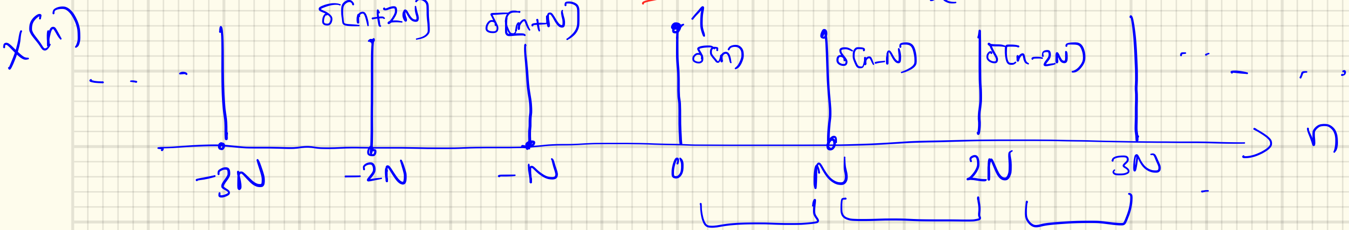
$\Rightarrow$ envelope is sinc function

plot the spectrum for $N = 10$, $L = 2$



See the periodic form.

Ex: Periodic Impulse Train: $x[n] = \delta_N[n] \triangleq \sum_{\ell=-\infty}^{\infty} \delta[n - \ell N]$



DT Fourier Series coeff of $x[n] = ?$

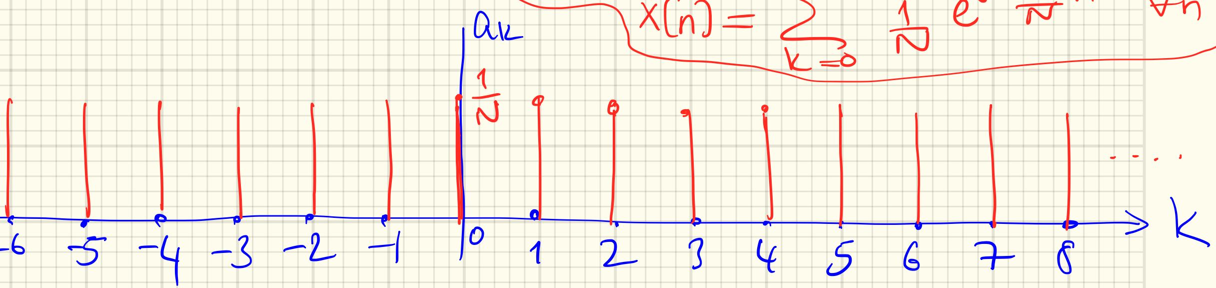
$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} \underbrace{x[n]}_{\delta[n]=1} \cdot \underbrace{e^{-j\frac{2\pi}{N}kn}} = \frac{1}{N} \underbrace{\sum_{n=0}^{N-1} e^{-j\frac{2\pi k}{N}n}}_{=1} = \boxed{\frac{1}{N}} = a_k$$

sum over 1 period

Use Fourier synthesis

$$x[n] = \sum_{k=0}^{N-1} \frac{1}{N} e^{j\frac{2\pi k}{N}n} \quad \forall n$$

Spectrum of $x[n]$



Chapter 5 SYSTEMS

: Maps an input (signal/data) to an output (signal)

$\approx$ Algorithms

CT Systems:

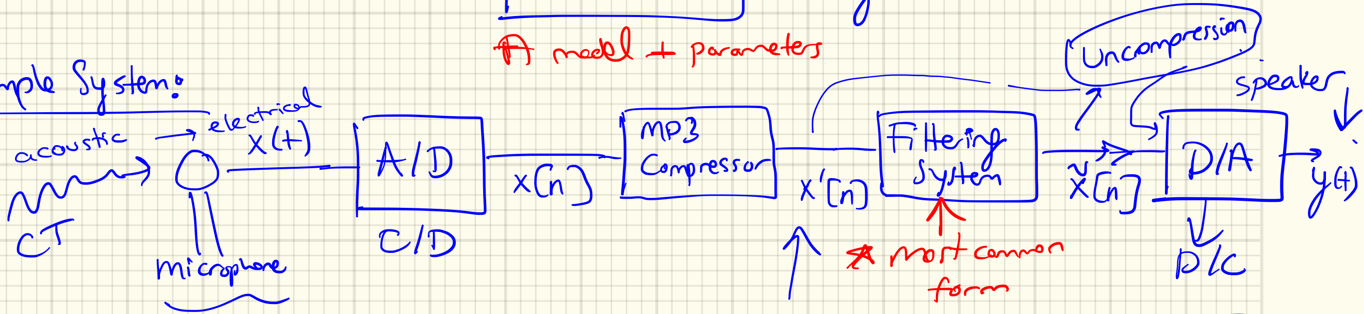
$x(t) \longrightarrow$ $\boxed{\text{CT System}}$ $\longrightarrow y(t)$

DT Systems

$x[n] \longrightarrow$ $\boxed{\text{DT System}}$ $\longrightarrow y[n]$

(A) model + parameters

Example System:



C/D
D/C.

we will cover these after we cover Fourier transform.

① Systems for removing unwanted features such as noise etc. from signal.

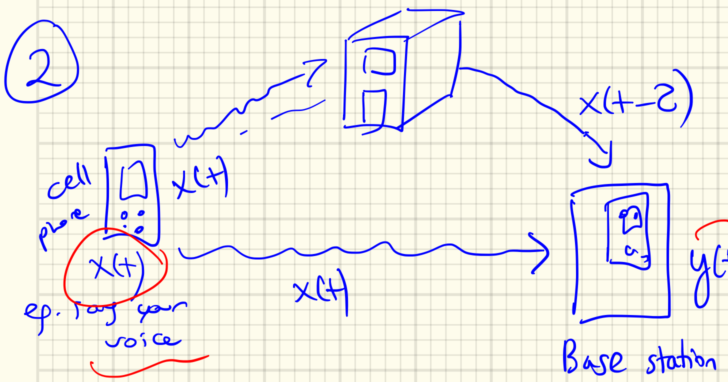
Filtering: $x[n] \longrightarrow$ Filter $\longrightarrow y[n]$

Input — Output

design the filter w/ your belief/assumption about the corruption or unneccesary components

We believe input is corrupted w/ noise
or input needs to be simplify

②



$$y(t) = x(t) + \underbrace{\alpha \cdot x(t-\tau)}_{\text{echo}}$$

received signal

insert

$$\boxed{\tilde{x} = y(t) - \alpha\, y(t-\tau) + \alpha^2 y(t-2\tau) - \alpha^3 y(t-3\tau)}$$

$$\boxed{\alpha << 1}$$

$$x(t) \longrightarrow \boxed{\text{System Transmitter}}_{H,\ \alpha} \longrightarrow y(t) \longrightarrow \boxed{\text{Inverse System}\ H^{-1}} \longrightarrow \tilde{x}(t) \approx x(t) - \cancel{\alpha^4 x(t-4\tau)}$$

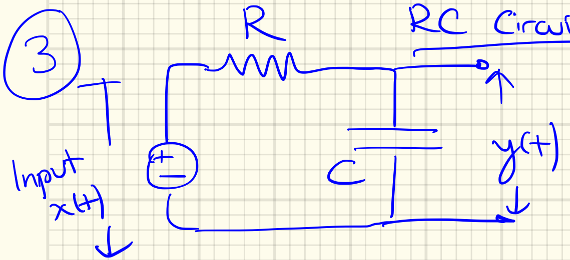
$\alpha^4 \approx 0$.

echo is killed.

→ This is an example of an inverse system; tries to eliminate unwanted parts ("echoes") in the signal.

Receive $y(t) \longrightarrow$ Inverse system $\longrightarrow \approx x(t)$

(3) **RC Circuit System**: CT System can be represented by a Differential Equations.



$$\frac{dy}{dt} + \frac{1}{RC}\, y(t) = \frac{1}{RC}\, x(t)$$

R, C: parameters of the system.

(4) NN: Neural Networks : Nonlinear System example

$\underline{x}[n] \longrightarrow$ NN system $\longrightarrow y[n] = g(\underline{\theta}\, g(\underline{\theta}(g(\underline{\theta}\underline{x}))))$

Input

Output

nonlinearity

parameters $\underline{\theta} = (w, b)$

Deep NNs: millions of parameters

Systems 1) model the effects of a physical phenomenon

2) Implement a desired effect on the signal (that we recorded)

e.g. ML Machine Learning : extract features from the data.

Discrete-time Systems : (DT)

DT Input Signal $x[n] \longrightarrow$ $\boxed{\text{DT System}}$ $\longrightarrow y[n]$

T : operator that models the system (mapping)

$$y[n] = T\left[x[n]\right]$$

maps the input x to the output y

models

$\boxed{\text{I/O (Input/output) relation}}$

Ex: $x[n] \longrightarrow \boxed{S} \longrightarrow y[n]$: I/O relation of the system S

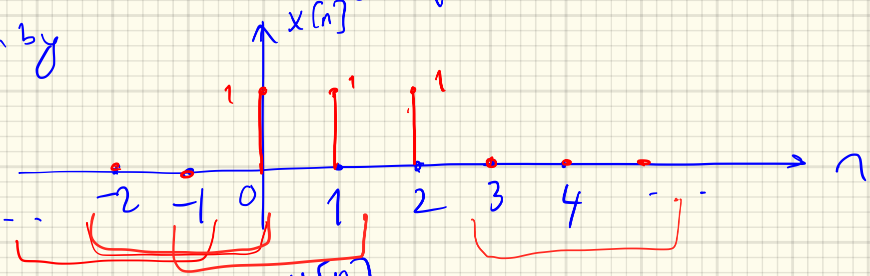
is given by

$$\boxed{y[n] = \frac{1}{3}x[n] + \frac{1}{3}x[n-1] + \frac{1}{3}x[n-2]}$$

3-pt Average System.

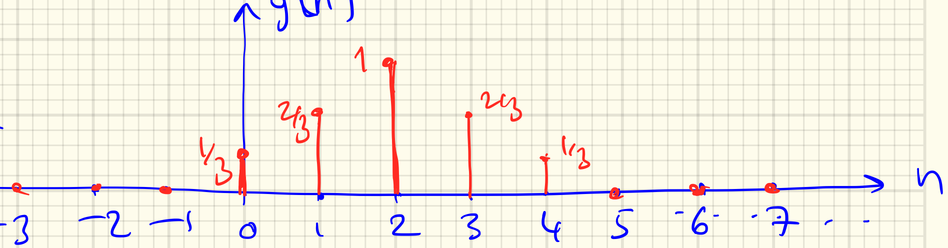
rule (map) that says output at time n is average of 3 consecutive inputs.

eg. say $x[n]$ is given by



$$y[0] = \frac{1}{3}\left(x[0] + x[-1] + x[-2]\right)$$

$$= \frac{1}{3}$$



$$y[1] = \frac{1}{3}\left[x[1] + x[0] + x[-1]\right) = \frac{2}{3}$$

$n=1$

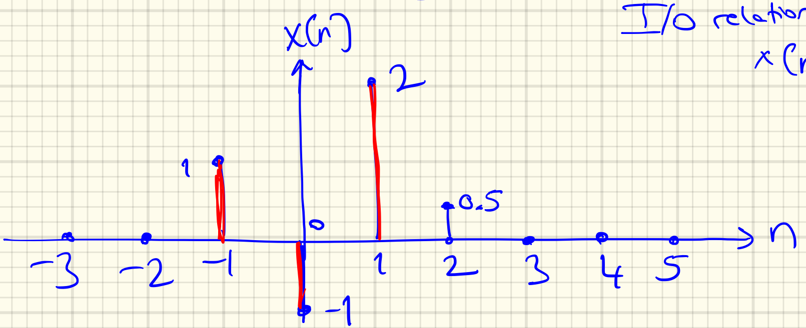
Ex: $y[n] = (x[n])^2$

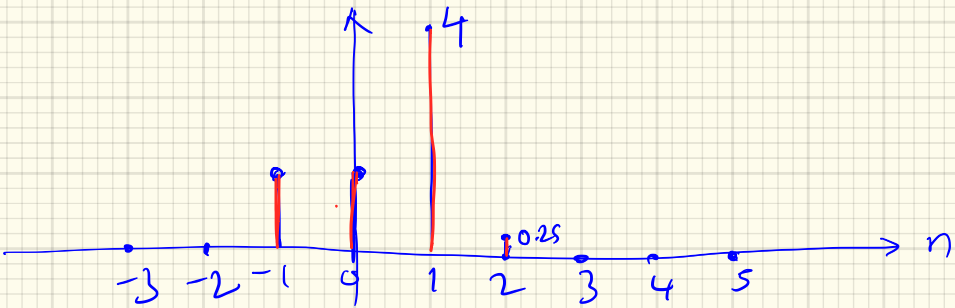
Squarer system

I/O relation of

$x[n] \longrightarrow \boxed{S} \longrightarrow y[n] = (x[n])^2$

eg.





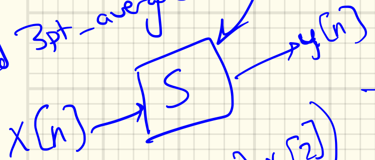
Note: The value of the output depends just at the value of the input at the same time point.

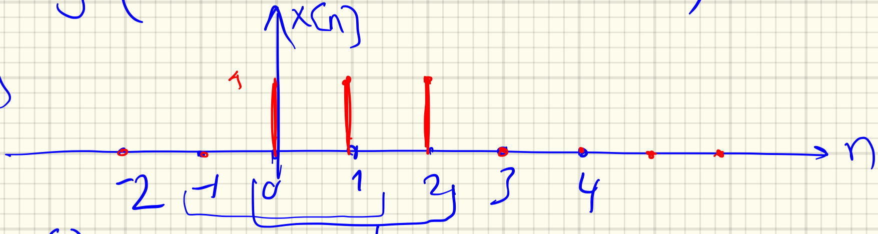
($\therefore$ Causal system)

Ex: $y[n] = \frac{1}{3}\left(x[n-1] + x[n] + x[n+1]\right)$

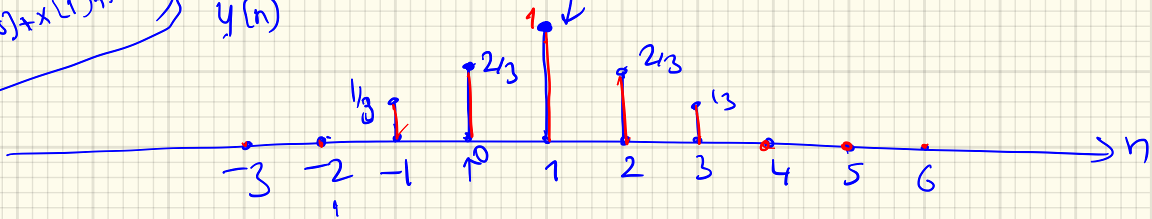
I/O relation

centered 3pt-averager





eg. $y[1] = \frac{1}{3}\left(x[0] + x[1] + x[2]\right)$



(System Property):

Causal System (Def): Output of the system depends on ONLY the current & past values of the input.

eg. $y[1]$: output at time $n=1$ → depends on $x[n]$ at $n=0$ ✓, $n=1$ ✓, $n=2$ ✗

∴ This past example (centered averager) is NOT a causal system.

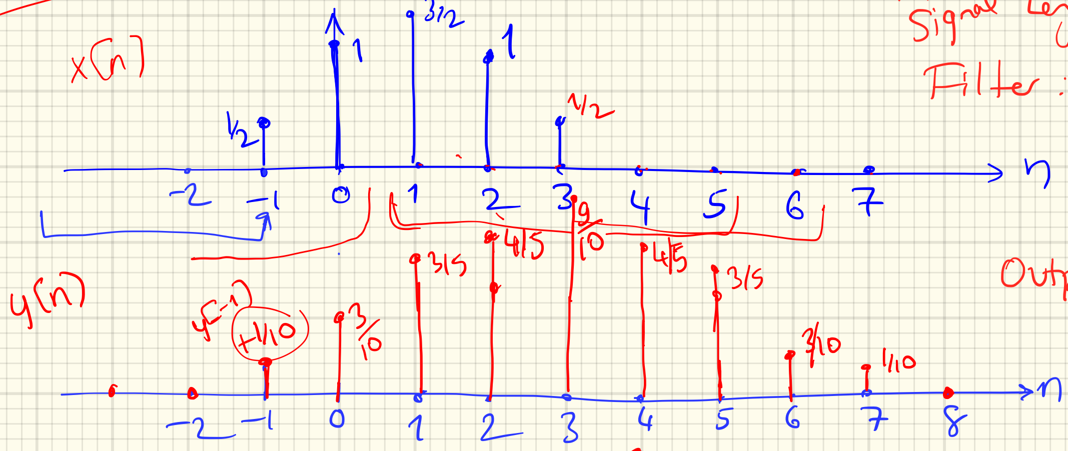
★ Causality is important for real-time ("online") implementation of a system.

Ex: $y[n] = \frac{1}{5}\left(x[n] + x[n-1] + x[n-2] + x[n-3] + x[n-4]\right)$

I/O relation for this system S

$x[n] \longrightarrow \boxed{S} \longrightarrow y[n]$

ex:



Input Signal Length = 5 = N

Filter: 5 = L

Output Length: 9 = N+L−1

5-pt Running Average Filter:

$y[4] = \frac{1}{5}\left[x[4] + x[3] + x[2] + x[1] + x[0]\right]$

$= \frac{1}{5}\left[0 + \frac{1}{2} + 1 + \frac{3}{2} + 1\right] = \frac{4}{5}$

exercise

fill in all y[n]

In general RAF (Running Average Filter) : (m+1)pt.

$$y[n] = \sum_{k=0}^{m} \underbrace{\boxed{\frac{1}{m+1}}}_{b_k} x[n-k]$$

b_k : filter weights

eg. 3 pt RAF

$$y[n] = \sum_{k=0}^{2} \frac{1}{3} x[n-k]$$

5pt RAF

$$y(n) = \sum_{k=0}^{4} \frac{1}{5} x[n-k]$$

Generalize to : FIR Filter (Finite impulse response)

$$\boxed{y(n) = \sum_{k=0}^{m} b_k \, x[n-k]}$$

FIR

b_k : filter coefficients

→ Linear Constant Coefficient Differential Equations

eg. $$\frac{d^4 y(t)}{dt^4} + a_3 \frac{d^3 y(t)}{dt^3} + a_2 \frac{d^2 y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_0 y_0(t) = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

CT

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Equivalent

DT

Linear Constant Coeff Difference Equations:

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]$$

I/O

$$x[n] \longrightarrow S\ (\mathrm{FIR}) \longrightarrow y(n)$$

$a_0 = 1$

$a_k = 0$, $k \neq 0$

$$y(n) = \sum_{k=0}^{M} b_k\, x[n-k]$$

A causal system

(FIR) Filter.

$$\rightarrow y[n] = b_0\, x[n] + b_1\, x[n-1] + \cdots + b_m\, x[n-m] \longrightarrow \{b_k\}_{k=0}^{m}$$

$$x[n] \longrightarrow \boxed{FIR} \longrightarrow y[n]$$

filter coeff.

<u>Ex:</u> $b_k = \{\underset{b_0}{+3},\ \underset{b_1}{-1},\ \underset{b_2}{2},\ \underset{b_3}{1}\}$

Q1: I/O relation of this system?

Q2. Causal system or not?

A1: $$y[n] = \sum_{k=0}^{3} b_k\, x[n-k] = 3x[n] - x[n-1] + 2x[n-2] + x[n-3]$$

A2: Causal system

Filter Length $= M+1 = L = 4$

Filter order $= M$: 3^{rd}-order filter.

Given $x[n]$ of length N (N-pt signal)

Q: Length of $y[n]$: $N+L-1$.

System Properties:

Causality ✓

① Linearity:

satisfies the property of superposition

i. Sum of Inputs $\longrightarrow$ $\boxed{\text{System } S}$ $\longrightarrow$ Sum of outputs

ii. Scale the input $\longrightarrow$ $\boxed{S}$ $\longrightarrow$ Doubled output (Scaled).

e.g. double the input

$$a\,x_1[n] + b\,x_2[n] \longrightarrow \boxed{S} \longrightarrow a\,y_1[n] + b\,y_2[n]$$

$$a.x_1[n] \longrightarrow \boxed{S} \longrightarrow a.y_1[n]$$

$$b.x_2[n] \longrightarrow \boxed{S} \longrightarrow b.y_2[n]$$

$$a\,y_1[n] + b\,y_2[n]$$

equal.

Testing for $\boxed{\text{Linearity}}$ of a System S:

I. $x_1[n] \longrightarrow \boxed{S} \longrightarrow y_1[n] \longrightarrow \otimes a$; $x_2[n] \longrightarrow \boxed{S} \longrightarrow y_2[n] \longrightarrow \otimes b$; $\longrightarrow \oplus \longrightarrow y'(n) = a\,y_1[n] + b\,y_2[n]$

II. $x_1[n] \longrightarrow \otimes a$; $x_2[n] \longrightarrow \otimes b$; $\longrightarrow \oplus \longrightarrow a\,x_1[n] + b.x_2[n] \longrightarrow \boxed{S} \longrightarrow y[n]$

Q. Is $y(n) = y'(n)$?

If yes → S is Linear

2 Time Invariance of a System : System behavior does not change w/ time.

Testing time Invariance:

I : $x[n] \rightarrow \boxed{S} \rightarrow y[n] \rightarrow \boxed{\text{Delay } n_0} \rightarrow y[n-n_0]$

II : $x[n] \rightarrow \boxed{\text{Delay } n_0} \rightarrow x[n-n_0] \rightarrow \boxed{S} \rightarrow z[n]$

equal ?

Is $z[n] == y[n-n_0]$ If yes, S is Time Inv.

No S is Not Time Inv.

Def (LTI) Linear Time Invariant Systems :

A system that is both Linear & Time Invariant is an LTI system.

Note : Def (Delay System) : $x[n] \rightarrow \boxed{\text{Delay } n_0} \rightarrow y[n] = x[n-n_0]$

↳ Causal ?

$x[n] \rightarrow \boxed{\text{Delay by } 2} \rightarrow x[n-2]$

$x[2] \rightarrow \quad \rightarrow x[0]$

causal $n_0 \geq 0$

not o/w.

$x[n] \rightarrow \boxed{\text{Delay by } n_0 = -2} \rightarrow x[n+2]$ output depends on a future value of the input.

$x[2] \rightarrow \boxed{\phantom{\text{Delay}}} \rightarrow x[4] \Rightarrow$

Not causal when $n_0 < 0$.

Ex: $y[n] = (x[n])^2$ Is this an LTI system?

(1) Time Inv?

I. $x[n] \rightarrow \boxed{S} \rightarrow (x[n])^2 = y[n] \rightarrow \boxed{\text{Delay}}_{n_0} \rightarrow (x[n-n_0])^2$

II $x[n] \rightarrow \boxed{\text{Delay}} \rightarrow x[n-n_0] \rightarrow \boxed{S} \rightarrow w[n]$

Yes Time Inv

(2) Linear?

I. $x_1[n] \rightarrow \boxed{S} \rightarrow (x_1[n])^2 \xrightarrow{\times a}$

$x_2[n] \rightarrow \boxed{S} \rightarrow (x_2[n])^2 \xrightarrow{\times b}$

$\oplus \rightarrow a y_1[n] + b y_2[n]$

$= a(x_1[n])^2 + b(x_2[n])^2$

II. $a x_1[n] + b x_2[n] \rightarrow \boxed{S} \rightarrow (a x_1[n] + b x_2[n])^2$

$= a x_1^2[n] + b x_2^2[n] + 2ab\, x_1[n] x_2[n]$ $\neq$

$\Rightarrow$ Not a Linear System $\therefore$ Not an LTI system.

Ex: $y[n] = x[-n]$ System's I/O relation is given.

$$x[n] \rightarrow \boxed{S} \rightarrow y[n]$$

Q. Is this an LTI system?

Linearity?

I. $x_1[n] \rightarrow \boxed{S} \rightarrow y_1[n] = x_1[-n] \xrightarrow{\times a}$

I. $x_2[n] \rightarrow \boxed{S} \rightarrow y_2[n] = x_2[-n] \xrightarrow{\times b}$

$\oplus \rightarrow a x_1[-n] + b x_2[-n]$

$=$ ✓

II. $a x_1[n] + b x_2[n] \rightarrow \boxed{S} \rightarrow a x_1[-n] + b x_2[-n]$

Yes, Linear System.

Time Inv?

I. $x[n] \rightarrow \boxed{S} \rightarrow x[-n] \rightarrow \boxed{\text{Delay } n_0} \rightarrow z[n] = x[-(n-n_0)]$

$= x[-n+n_0]$

II. $x[n] \rightarrow \boxed{\text{Delay } n_0} \rightarrow x[n-n_0] \rightarrow \boxed{S} \rightarrow y[n] = x[-n-n_0]$

system does a time-axis scaling by -1.

Not a Time-Invariant System

$\therefore$ S is not an LTI system.