BLG 354E Signals & Systems

Week 7

12.04.2021

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Recap : Systems

1) Model physical phenomena : → Inverse System Identification

2) Implement a desired effect on a signal. ← Filtering
   - → Time-Domain
   - → Frequency-Domain

- Filtering : removing unwanted parts / features in a signal.
- compression
- feature extraction

$x(t) \to \boxed{S} \to y(t)$

$x[n] \to \boxed{S} \to y(n)$

System S

LTI : Linear Time Inv. Systems.

Properties of System

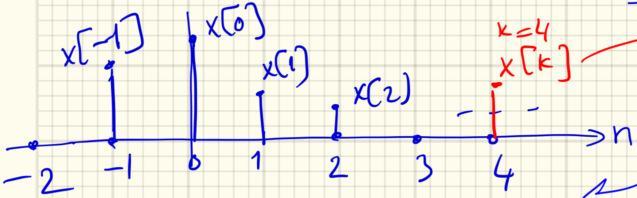
LTI
1) Linearity : S obeys superposition ✓
2) Time Inv : S : does not change behavior over time.

3) Causality : ✓ real-time / "online" implementation of systems.

4) Stability : today

LTI systems → have a simple I/O relationship depending impulse-response

For any signal: $x[n]$

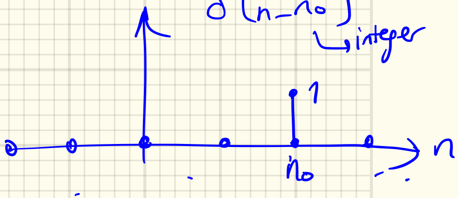


I can represent any DT signal $x[n]$

$$\boxed{x[n] = \sum_{k=-\infty}^{\infty} \underbrace{x[k]}\cdot\delta[n-k]}$$

$$x[n] = x[0]\,\delta[n] + x[1]\,\delta[n-1] + x[2]\,\delta[n-2] + \dots$$
$$+\, x[-1]\,\delta[n+1] + x[-2]\,\delta[n+2] + \dots$$
$$x[-1000]\,\delta[n+1000] + \dots$$

recall $\delta[n-n_0]$ ↳ integer



LTI system $\quad x[n] \longrightarrow \boxed{\text{LTI System } S} \longrightarrow y[n]$

T

$$y[n] = T\{x[n]\} = T\left\{ \underbrace{\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]} \right\} = \sum_{k} T\{x[k]\,\delta[n-k]\}$$

superposition of inputs

Use linearity of the system S

$$\rightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]\, \underbrace{T\{\delta[n-k]\}}$$

we use time invariance of LTI system S

$$\delta[n] \longrightarrow \boxed{S} \longrightarrow h[n]$$

$$\delta[n-k] \longrightarrow \boxed{S} \longrightarrow h[n-k]$$

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]}$$

Define impulse response

$$\boxed{h[n] \triangleq T\{\delta[n]\}}$$

$$\delta[n] \longrightarrow \boxed{\text{LTI System } S} \longrightarrow h[n]$$

$\delta[n]$ is input $\rightarrow T\{\delta[n]\} \rightarrow h[n]$

$h[n]$: impulse response of an LTI system.

this eq. defines Convolution operation

x[n] is convolved w/ h[n]

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \underbrace{x[n] * h[n]}$$

☆ For an LTI system S : h[n] : impulse response of the system S :

Given input $x[n] \longrightarrow \boxed{S} \longrightarrow \boxed{y[n] = x[n] * h[n]}$

h[n]   output.

Q: Is the FIR filter (system) LTI ?

exercise : show this

yes.

$$x[n] \rightarrow \boxed{S} \rightarrow y[n] = \sum_{k=-\infty}^{\infty} \underbrace{b_k}_{\triangleq\, h[k]} \, x[n-k]$$

FIR : $\{b_k\}$ filter coefficients

Properties of CONVOLUTION : $*$ ← convolution operator

① Commutative : $x_1[n] * x_2[n] = x_2[n] * x_1[n]$

show :

$$\sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k] = \sum_{m=-\infty}^{\infty} x_1[n-m]\, x_2[m] = x_2[n] * x_1[n]$$

$n-k=m \rightarrow$

$k = n-m$

② Associative : $x_1[n] * \left(x_2[n] * x_3[n]\right) = \left(x_1[n] * x_2[n]\right) * x_3[n]$

③ Distributive Over Addition: $x[n] * (h_1[n] + h_2[n])$

$$= x[n] * h_1[n] + x[n] * h_2[n]$$

④ Has an identity element ($*$ operator): $\delta[n]$: unit impulse sequence.

$$x[n] * \delta[n] = x[n]$$

show $\displaystyle\sum_{k=-\infty}^{\infty} \underline{x[k]}\, \underbrace{\delta[n-k]}_{n=k} = x[n] * \delta[n] = x[n]$

Q: $x[n] * \delta[n-1] = ?$

$$\sum_{k=-\infty}^{\infty} x[k]\,\underbrace{\delta[(n-1)-k]}_{k=n-1} = \underline{x[n-1]}$$

$$x[n] \longrightarrow \boxed{\delta[n-1]} \longrightarrow y[n] = x[n-1]$$

$h[n] =$

Delay System by 1 sample.

5

$$\delta[n-n_0] * x[n] = x[n-n_0]$$

Delay System by n_0: impulse response is $\delta[n-n_0]$ $= h[n]$.

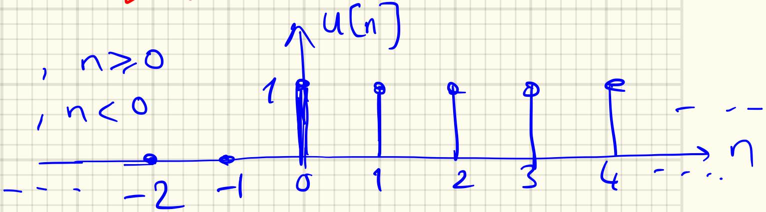
$$x(n) \longrightarrow \boxed{h[n]=\delta[n-n_0]} \longrightarrow x[n-n_0]$$

delayed input.

Impulse Response of an LTI system

Step Sequence:

$$u[n] = \begin{cases} 1 & , n \geq 0 \\ 0 & , n < 0 \end{cases}$$



$$u[n] = \delta(n) + \delta(n-1) + \delta(n-2) + \cdots$$

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k] \Rightarrow u[n] = \sum_{\ell=-\infty}^{n} \delta[\ell]$$

$\ell = n-k$ change of var.

STEP RESPONSE of an LTI System:

Set the input to a step seq.

$$u[n] \longrightarrow \underset{\text{LTI}}{\boxed{S}} \longrightarrow s[n]$$

$x(n) = u[n]$ $\qquad$ $y[n]$

LTI

$$u[n] \longrightarrow \boxed{S} \longrightarrow s[n] = y[n] = h[n] * u[n]$$

$$\boxed{S} \rightarrow h[n]$$

$$x[n] = \sum_{k=0}^{\infty} \delta[n-k] \longrightarrow \boxed{S} \longrightarrow y[n] = h[n] * x[n]$$

$$y[n] = h[n] * \left(\sum_{k=0}^{\infty} \delta[n-k] \right) = \sum_{k=0}^{\infty} \underbrace{\left(h[n] * \delta[n-k] \right)}_{h[n-k]}$$

$*$: distrib. over addition

$\rightarrow$ Step response: $s[n] = y[n] = \sum_{k=0}^{\infty} h[n-k]$

$*$ Note: another property of convolution.

$$\delta[n] * \delta[n-n_0] = \delta[n-n_0]$$

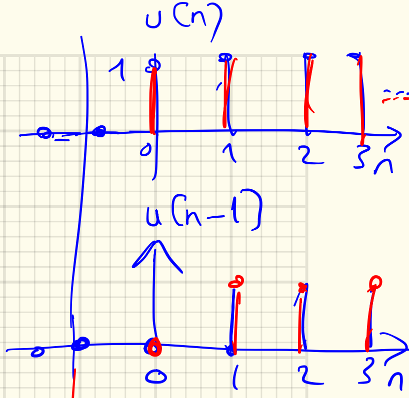
$$\delta[n-n_1] * \delta[n-n_0] = \delta[n-(n_1+n_0)]$$

Ex: Given step response $s(n) \rightarrow h(n) = ?$

① $u[n] \longrightarrow \boxed{S} \longrightarrow s[n]$

Q: $\delta(n) \longrightarrow \boxed{S} \longrightarrow h(n) = ?$

write $\delta[n]$ i.t.o. $u(n)$ $\quad \delta(n) = u[n] - u[n-1]$



$\delta(n) = u(n) - u(n-1)$

use superposition of inputs.

Time Inv.

$u(n-1) \longrightarrow \boxed{S} \longrightarrow s(n-1)$

Linearity.

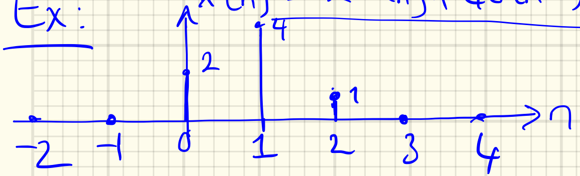
$\underbrace{u[n] - u(n-1)}_{} = \underbrace{\delta(n)}_{} \longrightarrow \boxed{S} \longrightarrow h(n) = s(n) - s[n-1]$

Computation of Convolution:

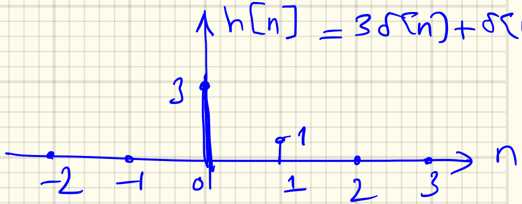
$x[n] \longrightarrow \boxed{\begin{matrix} S \\ h[n] \end{matrix}} \longrightarrow x[n] * h[n] = y[n]$

$\equiv$ Given $x[n]$ (input) & $h[n]$ (impulse response) $\rightarrow$ compute $y[n]$.

Ex: $x[n] = 2\delta[n] + 4\delta[n-1] + \delta[n-2]$



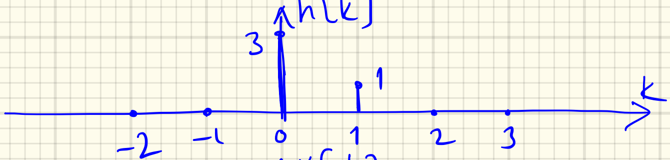
$h[n] = 3\delta[n] + \delta[n-1]$



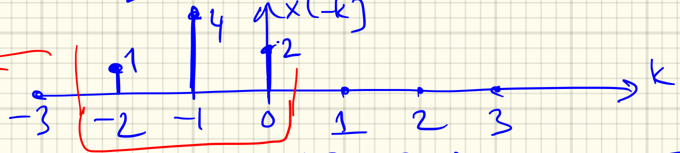
$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

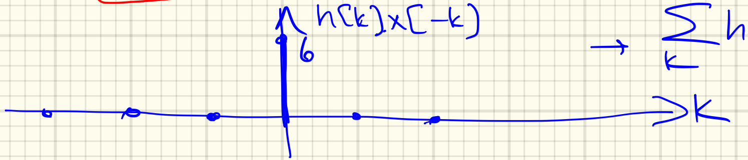
$$y(0) = \sum_{k=-\infty}^{\infty} \underbrace{h(k)}\, \underbrace{x(-k)}$$

$$y(1) = \sum_{k} \underbrace{h(k)}\, \underbrace{x(1-k)}$$



flipped filter

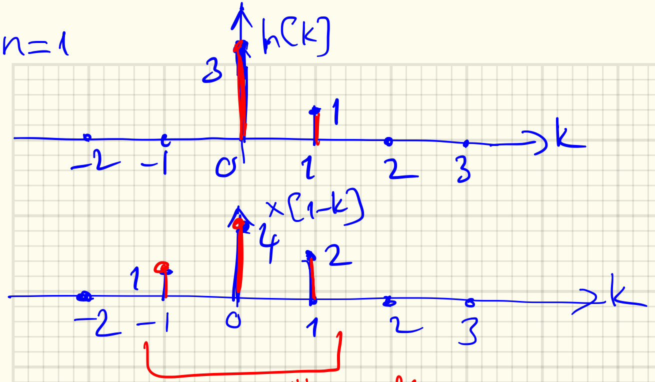




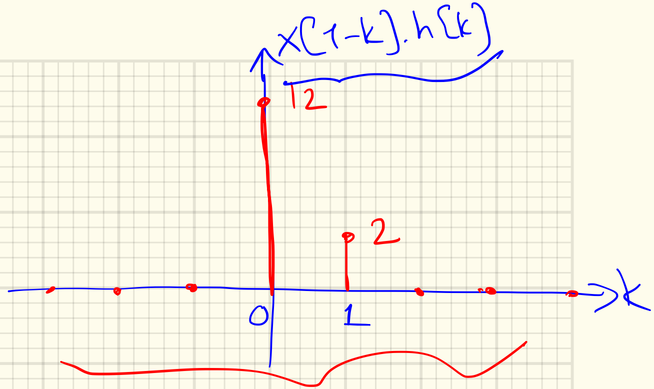
$$\rightarrow \sum_{k} h(k)x(-k) = 6 = y[n=0]$$

$$\boxed{y(0) = 6}$$

slide this flipped filter signal over the other signal n=1



filter shift

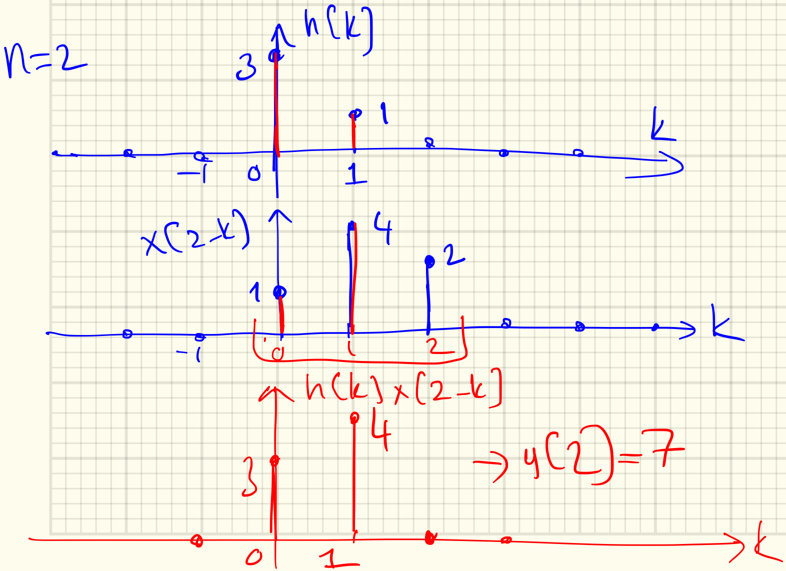


sum (integrate) over k.

$$y(1)=\sum_{k} x[n-k]h[k] = \boxed{14}$$

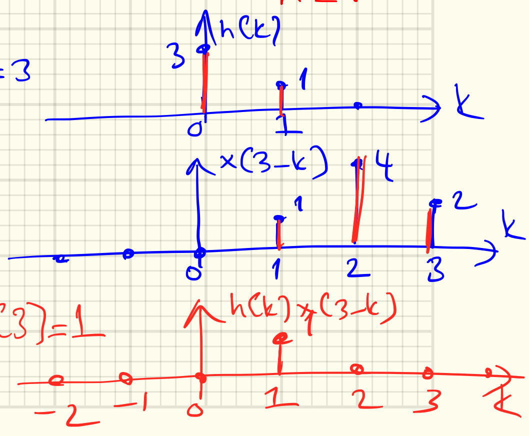
n=1

n=2



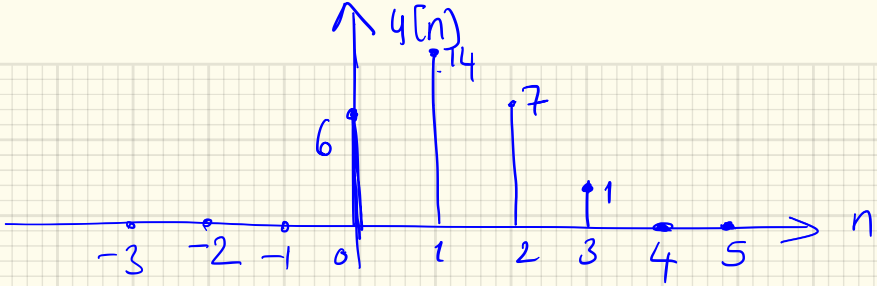
$\rightarrow y(2)=7$

n=3



y[3]=1

Result $\Rightarrow$



Due to commutativity of convolution:

→ We would arrive at the same result if we worked w/

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

Another way for DT convolution:

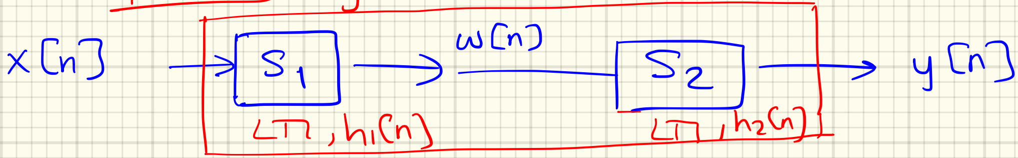
$$y[n] = \left(x[n] = 2\delta[n] + 4\delta[n-1] + \delta[n-2]\right) * \left(h[n] = 3\delta[n] + \delta[n-1]\right)$$

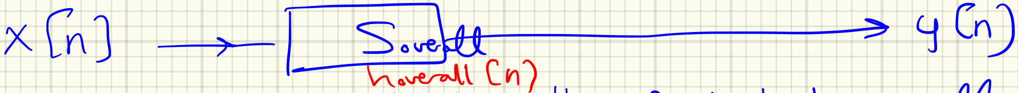
$$= 6\delta[n] + 12\delta[n-1] + 3\delta[n-2] + 2\delta[n-1] + 4\delta[n-2] + \delta[n-3]$$

$$y[n] = 6\delta[n] + 14\delta[n-1] + 7\delta[n-2] + \delta[n-3]$$

The same result as in the 1st way of convolution: using a graph method.

Cascaded LTI Systems:





Q: What is the equivalent overall system?

$$y[n] = h_2[n] * \overbrace{(x[n] * h_1[n])}^{w[n]} = h_2[n] * (h_1[n] * x[n])$$

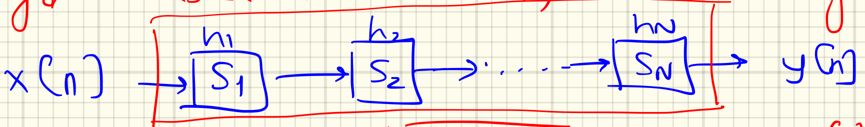
(commutative)

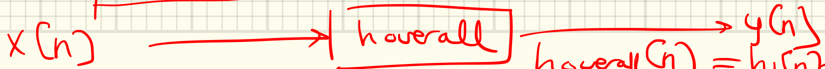
$$y[n] \underset{\text{(associative)}}{=} \underbrace{(h_2[n] * h_1[n])}_{h_{overall}[n]} * x[n]$$

$$h_{overall}[n] = h_1[n] * h_2[n]$$

Overall system impulse response for Cascaded (in series) connection of LTI systems

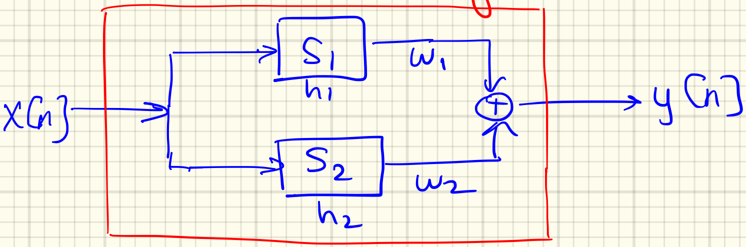
Valid For N LTI systems





$$h_{overall}[n] = h_1[n] * \cdots * h_N[n]$$

Parallel Connection of LTI Systems :



$h[n]$: Overall system S : $h(n) = ?$ i.t.o h_1 & h_2

$$y[n] = \big(\overbrace{x(n) * h_1[n]}^{w_1(n)}\big) + \big(\overbrace{x(n) * h_2[n]}^{w_2(n)}\big)$$

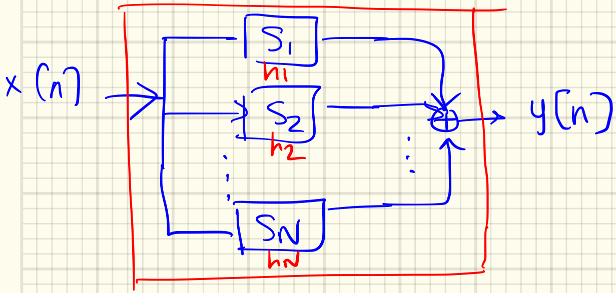
$$= x[n] * \underbrace{(h_1[n] + h_2[n])}_{h[n]}$$

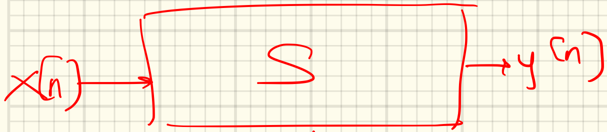
$$y[n] = x(n) * h[n]$$

Overall system in Parallel connection has : $h(n) = h_1[n] + h_2(n)$ (addition)

" " Cascaded " " : $h[n] = h_1[n] * h_2[n]$ (convolution)

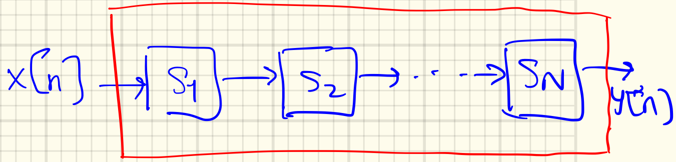
Extension to N systems:

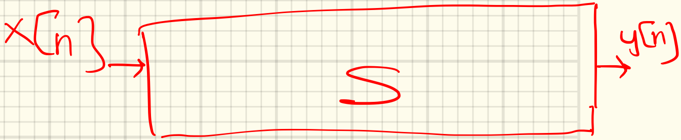




$$h[n] = \sum_{k=1}^{N} h_k[n]$$: Addition of all impulse responses

Parallel System:



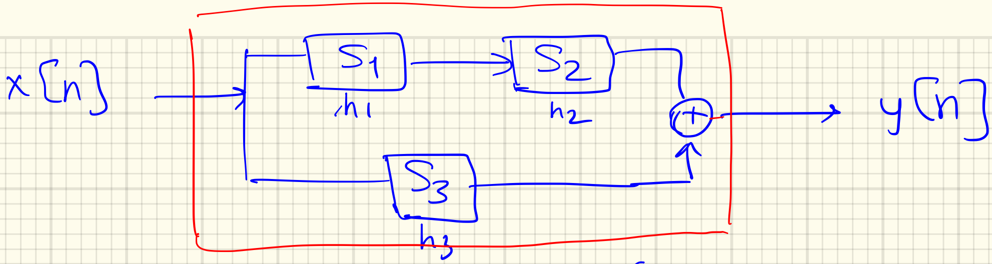


$$h[n] = \underbrace{h_1[n] * h_2[n] * \cdots * h_N[n]}$$

Cascaded System:

Convolution of all impulse responses

Ex:



$$h_{overall}[n] = ?\ \left(h_1[n] * h_2[n]\right) + h_3[n]$$

You can think of many different combinations and calculate their overall system response.

Recall Causality of a System:

$\boxed{\text{General Defn}}$: A system is causal if the output $y(n)$ only depends on current & past values of $x(n)$.

ex: $y(n) = x[n] + (x[n-1])^2$; Is this system causal?

I/O relation of a system

$$\underbrace{y[2]}_{n=2} = x[2] + (x[1])^2$$ Yes ✓

Checking Causality for LTI Systems: → check the impulse response.

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

for $k \geq 0$ $y[n]$ → depends $x(n)$ & $\underbrace{x[n-k]}_{\text{past values}}$ ✓

for $k < 0$ $x[n+k']$: future values of $x(n)$.

Def (Causality for LTI): An LTI system is causal iff (if and only if) $h[n] = 0$ for $n < 0$.

Recall General system of Const. Linear Coeff. Difference Equations

→ an important class of LTI systems:

I/O relation: $$\sum_{k=0}^{N} a_k \, y[n-k] = \sum_{k=0}^{M} b_k \, x[n-k]$$

$N = 0 \to a_0 = 1$: $$\boxed{y[n] = \sum_{k=0}^{M} \underbrace{b_k}_{h[k]} x[n-k]}$$

FIR System → Finite # b_k, $h(k)$

ex FIR filter: Let $x(n) = \delta(n)$

↳ $h(n) = \frac{1}{2}\delta(n-1) - \frac{3}{2}\delta(n-1)$ ⟷ ex: $y[n] = \frac{1}{2}x[n] - \frac{3}{2}x[n-1]$ → $b_0 = \frac{1}{2} = h[0]$

$b_1 = -\frac{3}{2} = h[1]$

Extra: Infinite Impulse Response (IIR):

I/O: $y[n] - \frac{1}{2}y[n-1] = x[n] \to y[n] = \frac{1}{2}y[n-1] + x[n]$

Find impulse response: Let $x[n] = \delta[n]$

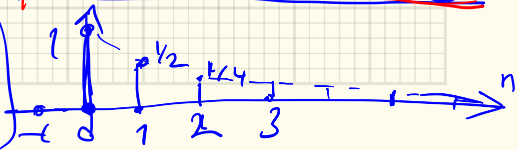
System is at rest $h[-1] = 0$, $y[-1] = 0$

$h[0] = \underset{h(-1)}{0} + x[0] = 1 = y[0]$

$h(1) = \frac{1}{2}\underbrace{h[0]}_{1} + \underbrace{x[1]}_{0} = \frac{1}{2}$

$h[2] = \frac{1}{2}h[1] + x(2) = \frac{1}{4}$;

ex: $$\boxed{h[n] = \left(\frac{1}{2}\right)^n \cdot u[n]}$$



Ex: Is this LTI system causal? $h[n] = \left(\frac{1}{2}\right)^n u[n]$

Check whether $h[n] = 0$ for $n < 0$? ✓

This IIR filter is causal.

4th property of Systems $\boxed{\text{STABILITY}}$

(General defn.: robustness against perturbations)

Def: (Bounded Input Bounded Output $\equiv$ BIBO) Stability:

Def: System S is stable iff every bounded input produces a bounded output.

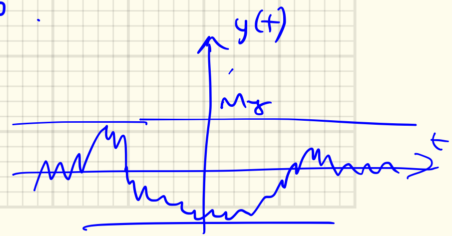
In math terms: If $|x[n]| \leq M_x$, $M_x < \infty$.

($x(t)$)

then $|y[n]| \leq M_y$, $M_y < \infty$.

($|y(t)|$)

BIBO $\Longrightarrow$



$x[n] \longrightarrow \boxed{S} \xrightarrow{h[n]} y[n]$

If the system is $\boxed{LTI}$: Stability of S is defined ito the impulse response

Let $|x[n]| \leq M_x$

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] \right|$$

$$\leq \sum_{k} |h[k] . x[n-k]|$$

$$\leq \sum_{k} |h[k]| . \underbrace{|x[n-k]|}$$

$$\leq \underbrace{\sum_{k} |h[k]|}_{\text{if this is bounded}} . \underline{M_x} \quad \text{a const.}$$

$$|y[n]| \leq M_y , \quad M_y = M_x \sum |h[k]|$$

Def (Stability for LTI system): LTI system is stable iff $\boxed{\sum_{k=-\infty}^{\infty} |h[k]| < \infty}$

Ex: $y[n] = x[-n]$ → How would you check (BIBO) stability?

Is this system LTI? No.

Let $|x[n]| < M_x$, $M_x < \infty$. Stable.

$|y[n]| = |x[-n]| < M_x$

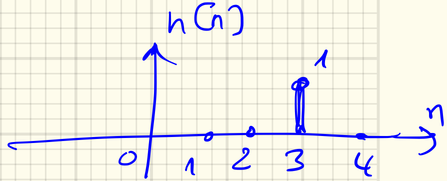
Ex: $y[n] = \dfrac{1}{x[n]}$: Is this LTI? No

Is this stable? No.

Let $x[n] = 0$ a finite $y[n] \nearrow \infty$. Not a stable system.

Ex: $y[n] = x[n-3]$: Delay by 3 system.

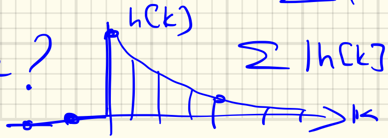
LTI system : $h[n] = \delta[n-3]$



Stable

$\sum |h[n]| = 1.$

Ex: $h[n] = \left(\frac{1}{3}\right)^n u[n]$: Is this stable?



$\sum |h[k]| < \infty$

Stable

IIR:

Ex: Given $h[n] = \delta[n-2]$ & Output $y[n] = u[n-3] - u[n-6]$ } Input ? $x[n]$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

Delay by 2 system

$$x[n] = u[n-1] - u[n-4]$$

Ex: Given LTI system: $u[n] \longrightarrow \boxed{S} \longrightarrow \delta[n] + 2\delta[n-1] - \delta[n-2]$

$x[n] = 3u[n] - 2u[n-4] \longrightarrow \boxed{S} \longrightarrow$? $y[n] = ?$

Use LTI properties

$$3u[n] \longrightarrow \boxed{S} \longrightarrow 3\delta[n] + 6\delta[n-1] - 3\delta[n-2]$$

$$-2u[n-4] \longrightarrow \boxed{S} \longrightarrow -2\delta[n-4] - 4\delta[n-5] + 2\delta[n-6]$$

$+$

$$x[n] \longrightarrow \boxed{S} \longrightarrow 3\delta[n] + 6\delta[n-1] - 3\delta[n-2] - 2\delta[n-4] - 4\delta[n-5] + 2\delta[n-6]$$

Homework exercise:

$$y[n] = \sum_{k=0}^{6} B^k \, x[n-k] \quad , \quad B \text{ a real number.}$$

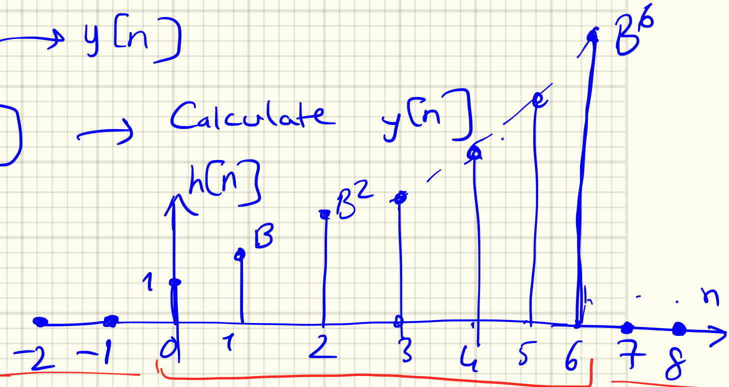
I/O

$$x[n] \longrightarrow \boxed{S} \longrightarrow y[n]$$

Let $x[n] = \delta[n] - B\,\delta[n-1] \longrightarrow$ Calculate $y[n]$

$h[n] = ?$ Calculate h.

FIR filter.



$\hookrightarrow h[n] = B^n \underbrace{[u[n] - u[n-7]]}$

Result. $y[n] = \cancel{B^0}\,\delta[n] - B^7 \delta[n-7]$
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