BLG 354E Signals & Systems

19.04.2021

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Wrap-up $\boxed{\text{DT Systems}}$ (LTI) → this course

Recall $\boxed{\text{First-order Difference FIR filter}}$ (finite no of coefficients) : → detects abrupt changes in a signal.

$$y[n] = x[n] - x[n-1]$$

$$h[n] = \delta[n] - \delta[n-1]$$

$$x[n] \longrightarrow \boxed{LTI} \longrightarrow y[n]$$

$h[n]$ → filter

characterized by impulse response

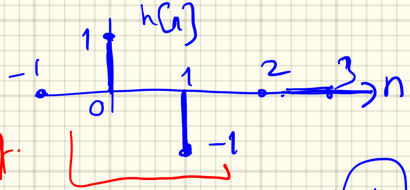
$*$: convolution

$$y(n) = x[n] * h[n]$$

Impulse response of the system

FIR

2 filter coeff.

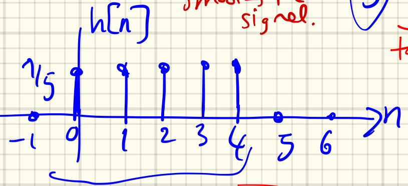


Compare to $\boxed{\text{5-pt RAF}}$: $y(n) = \frac{1}{5} \underbrace{\sum_{k=0}^{4} x(n-k)}_{\text{takes sums}}$ → $h(n) = \frac{1}{5} \sum_{k=0}^{4} \delta[n-k]$

Smooths the signal.

FIR

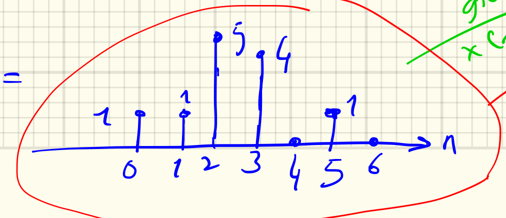
5 filter coefficients



exercise

Calculate the output $y(n)$ for both filters

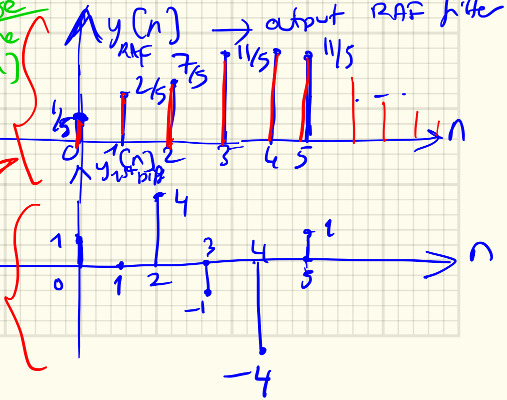
Let $x(n) =$

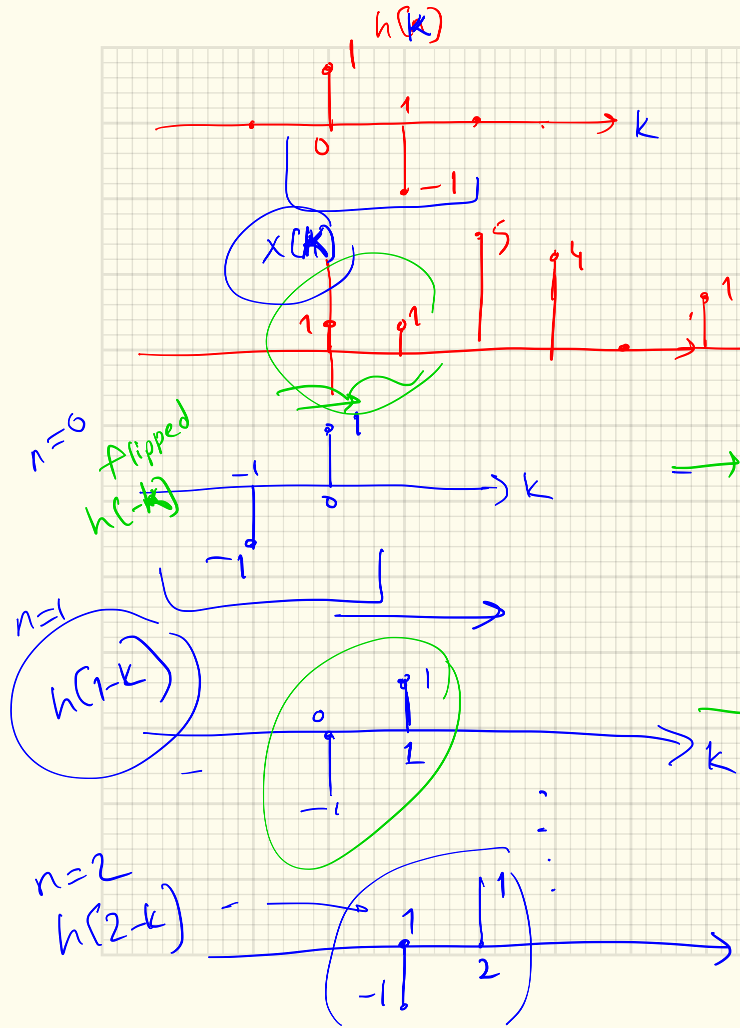


given $x(n)$

gets smoothed.

?





$\to y[0]=1$

$\to y(1)=0 \quad : \sum_k x[k]\cdot h[1-k]$

$y[2]=\sum_k x[k]\, h(2-k) = 4$

Mechanics of $\boxed{\text{convolution}}$ $\overset{\text{almost}}{=}$

$\boxed{\text{correlation}}$

is like convolution without flipping the filter.

$\boxed{\text{CNNs}} \longleftrightarrow$ correlations

$\boxed{\text{fft}} \to$ in freq. domain. we'll see later.

Continuous-time (CT) Systems (SP First Chapter 9)

$x(t) \longrightarrow$ CT System $\longrightarrow y(t)$

CT signal input — CT signal output

?

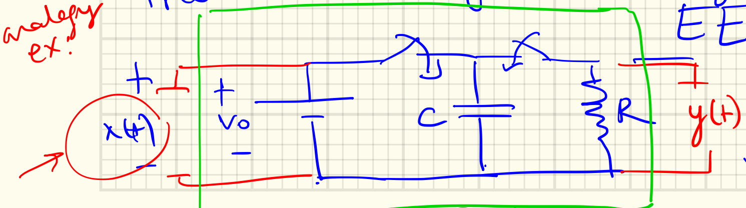
sound waves captured via microphone $x(t) \longrightarrow$ CT System $\longrightarrow y(t)$

$x(nT_s)$, T_s: A/D $\longrightarrow x[n] \longrightarrow$ DT system processing $\longrightarrow y[n] \longrightarrow$ D/A $\longrightarrow y(t)$

Digital Signal Processing

There are also of course analog systems like Circuits

analogy ex.!



CT System

EE Eng. / Mech. Eng. Spring (Suspension Systems.

Completely CT systems.

$\Rightarrow$ Differential eqn to model the system.

Impulse Response of the system

$\boxed{\delta(t)} \rightarrow \boxed{\text{CT System}} \rightarrow h(t)$: impulse response

Input: CT impulse fn. $\rightarrow$ measure the response of (LTI) CT system against the input $\delta(t)$.

$\equiv$ perturbing the system w/ a ∞ly short duration signal.

LTI

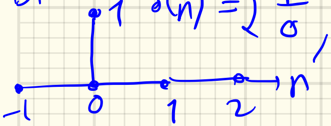
$x(t) \rightarrow \boxed{\text{CT System}} \rightarrow y(t) = h(t) * x(t)$

LTI, $h(t)$

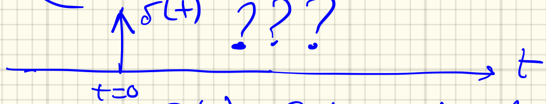
$*$: CT convolution

Recall DT

$$\delta[n] = \begin{cases} 1 & n=0 \\ 0, & n \neq 0 \end{cases}$$



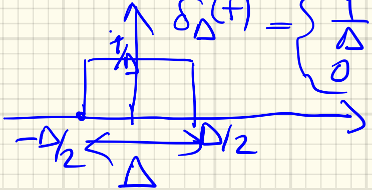
$\cancel{\delta(t)}$: not a function $\rightarrow$ generalized function.

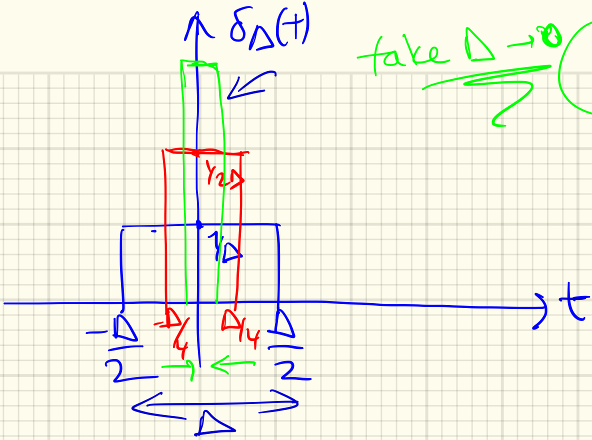


$\delta(t)$???

We will define it in the limit.

$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & -\frac{\Delta}{2} \leq t \leq \frac{\Delta}{2} \\ 0, & \text{else} \end{cases}$$





take $\Delta \to 0$

$\boxed{\Delta \to 0}$

$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & |t \leq \frac{\Delta}{2}| \\ 0, & o/w \end{cases}$$

$$\delta(t) \triangleq \lim_{\Delta \to 0} \delta_\Delta(t)$$

symbol ≡ defined as

$$\int_{-\infty}^{\infty} \delta_\Delta(t)\,dt = 1 \quad \text{Area underneath} = 1$$

$\Delta \to 0$

$$\boxed{\int_{-\infty}^{\infty} \delta(t)\,dt = 1} \quad \text{Area underneath} = 1$$

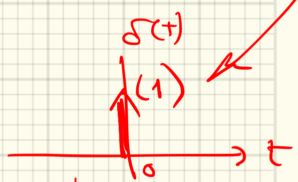
In the end, $\delta(t)$ is an ∞ amp but ∞-ly short duration signal $\rightarrow$ $\delta(t)$ represents an instantaneous perturbation signal

$\boxed{\text{Properties of } \delta(t)}$:

1) $\delta(t) = 0, \quad t \neq 0$

2) $\int_{-\infty}^{\infty} \delta(t)\,dt = 1 \equiv \int_a^b \delta(t)\,dt = 1$

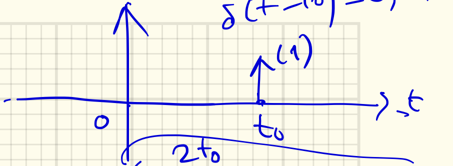
$a < 0 < b$.



But we cannot say ~~$\delta(t) = 1$~~ at $t = 0$

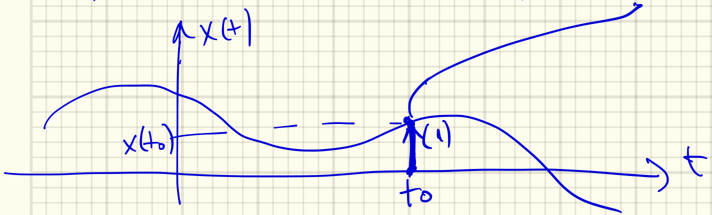
3) Sampling Property:

$\delta(t-t_0)=0,\ t\neq t_0$



$$\int_{0}^{2t_0}\delta(t-t_0)\,dt = 1$$

i) $x(t)\cdot\delta(t-t_0) = x(t_0)\cdot\delta(t-t_0)$



ii) $$\int_{-\infty}^{\infty}\underbrace{x(t)\delta(t-t_0)}_{x(t_0)\delta(t-t_0)}dt = x(t_0)\cdot\underbrace{\int_{-\infty}^{\infty}\delta(t-t_0)\,dt}_{1} = x(t_0)$$

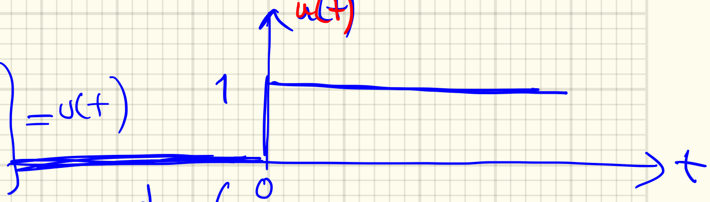
4) Relation btw u(t) & δ(t)

step fn.

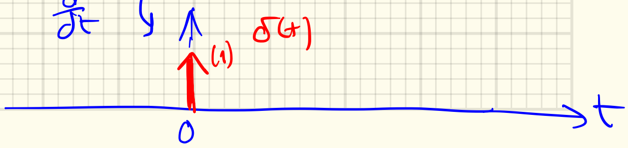
Look at $$\underbrace{\int_{-\infty}^{t}\delta(\tau)\,d\tau}_{} = \begin{cases} 1, & t\geq 0 \\ 0, & t<0 \end{cases} = u(t)$$

Integral of the impulse fn. is step fn.

$$\Longrightarrow \frac{du(t)}{dt} = \delta(t)$$

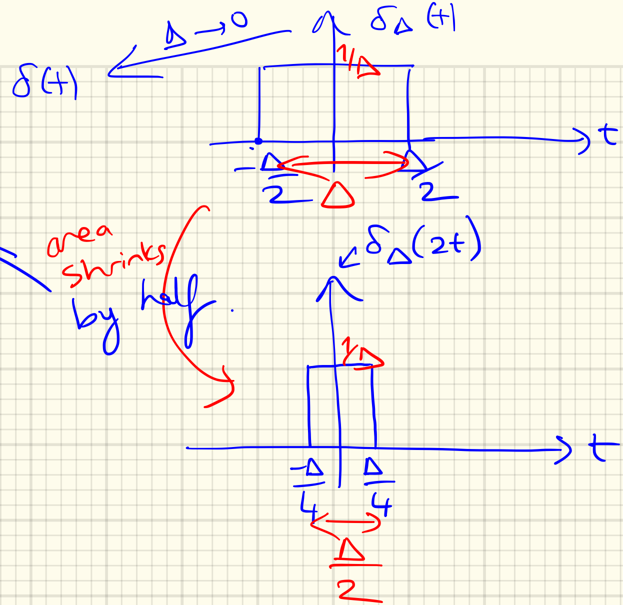


$\frac{d}{dt}$



⑤ Time-scaling $\delta(t)$:

say $$\boxed{\delta(2t) = \frac{1}{2}\delta(t)}$$



Generalize to

$$\boxed{\delta(at) = \frac{1}{|a|}\delta(t)}$$

e.g. $\delta(-t) = \delta(t)$

⑥ We can write any signal

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

τ : continuous variable.

integral version

DT: $$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

k : integer index

CT System Examples :

$$x(t) \longrightarrow \boxed{S} \longrightarrow y(t) = T\{x(t)\}$$

eg. $y(t) = (x(t))^2$ Squarer system

ex: $y(t) = x(t-t_0)$: Delay system

$$x(t) \longrightarrow \overset{S}{\boxed{\text{Delay System}}} \longrightarrow y(t) = x(t-t_0)$$

$\underline{LTI}$. t_0

S T

Impulse response : Let $x(t)=\delta(t) \longrightarrow \boxed{S} \longrightarrow h(t) = \delta(t-t_0)$.

ex: Differentiator System: $y(t) = \frac{d}{dt}x(t)$

1st order

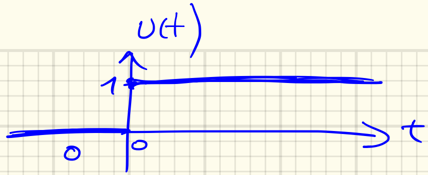
Integrator System : $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$

$$x(t) \longrightarrow \boxed{\int} \longrightarrow y(t)$$

(impulse response ?)

Integrator impulse response

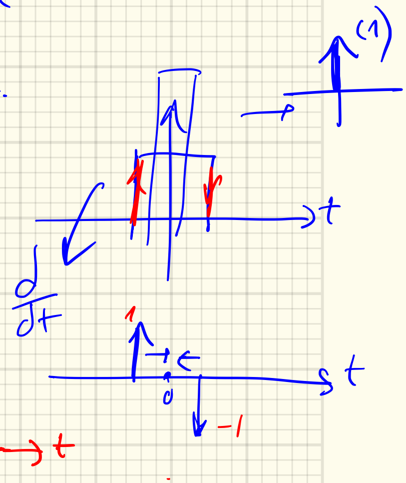
$$h(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau \longrightarrow$$



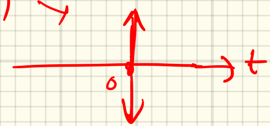
$= u(t)$ is the impulse response of the Integrator.

Ex: Differentiator System: $y(t) = \frac{d}{dt} x(t)$

Let $x(t) = \delta(t) \Rightarrow h(t) = \frac{d}{dt}\delta(t)$



$\triangleq \delta^{(1)}(t)$



Ex: Modulator System:

$$y(t) = x(t) \cdot \underline{\cos \omega_0 t}$$

high frequency sinusoidal.

CT Systems: Properties

$\boxed{\text{Linearity} \quad \text{Time Invariance}}$) LTI.

Causality

Stability

) $\equiv$ DT System properties.

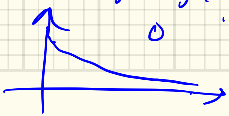
Ex: Let $h(t) = e^{-3t}u(t)$ for an LTI system. Is it stable?

causal ✓

System	Stability	Causality
General	BIBO	System Only uses current & past values of the input
LTI	$\int_{-\infty}^{\infty} \lvert h(t) \rvert dt < \infty$	$h(t) = 0$ for $t<0$

$h(t)$ or $h[n]$ characterizes the system.

absolute integrability (CT) $\equiv$ (summability DT)

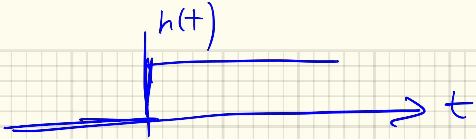


eg. $\int_0^{\infty} \lvert e^{-3t} \rvert \, dt = \frac{(-1)}{3} e^{-3t} \Big|_0^{\infty}$

$= -\frac{1}{3}(-1+0) = \frac{1}{3}$

stable

Ex: Integrator system: $h(t) = u(t)$



Causality: Yes.

Stability: $\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{0}^{\infty} 1 \cdot dt \not< \infty$. Unstable.

LTI: Linearity, Time-Inv. } show them as the same way we did in DT system.

For an LTI system: I/O relation is given by:

$$\boxed{\begin{aligned} y(t) &= h(t) * x(t) \\ y(t) &= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \end{aligned}}$$

Convolution Integral (similar to convolution sum $\sum h[k]\, x[n-k]$)

k: dummy variable.

τ: Integration variable

Properties of CT Convolution!

* Same as before as in DT Convolution.

1) Commutative: $x(t) * h(t) = h(t) * x(t)$

$$\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{\infty}^{-\infty} x(t-t')\, h(t')\, (-dt') = \int_{-\infty}^{\infty} h(t')\, x(t-t')\, dt'$$

show by: change of variables

$t' = t-\tau \rightarrow d\tau = -dt'$

$\tau = t-t'$

2) Associative: $x(t) * \big(h_1(t) * h_2(t)\big) = \big(x(t) * h_1(t)\big) * h_2(t)$

3) Distributive Over Addition:

$x(t) * \big(x_1(t) + x_2(t)\big) = x(t) * x_1(t) + x(t) * x_2(t)$

4) Identity Element: $\delta(t)$:

$\underline{x(t) * \delta(t) = x(t)} = \delta(t) * x(t)$

5) $\delta(t-t_0) * x(t) = x(t-t_0) \Rightarrow \int \underbrace{x(\tau)\, \delta(t-t_0-\tau)}_{\text{samples} \rightarrow \tau = t-t_0}\, d\tau = x(t-t_0)$

$\rightarrow \delta(t-t_0)$ impulse response of the Delay system.

$$x(t) \longrightarrow \boxed{\delta(t-t_0) = h} \longrightarrow x(t-t_0) = x(t) * \delta(t-t_0)$$

S : Delay system

$$\delta(t-t_1) \longrightarrow \boxed{} \longrightarrow y(t) = \delta(t-t_1) * \delta(t-t_0)$$

$$= \delta(t - t_1 - t_0)$$

$$= \delta(t - (t_1 + t_0))$$

Ex: Calculate $y(t) = \underbrace{u(t)}_{x(t)} * \underbrace{u(t)}_{h(t)}$ CT convolution.

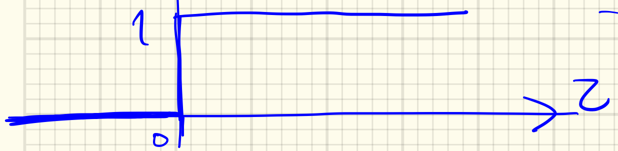
$h(t) \rightarrow$ integrator.

$$y(t) = \int_{-\infty}^{\infty} \underbrace{u(\tau)}_{= \begin{cases} 1, & \tau \geq 0 \\ 0, & \tau < 0 \end{cases}} u(t-\tau)\, d\tau = \int_{0}^{\infty} 1 \cdot \underbrace{u(t-\tau)}_{= \begin{cases} 1, & \overbrace{t-\tau \geq 0}^{= \tau \leq t} \\ 0, & o/w \; = \tau > t \end{cases}} d\tau$$

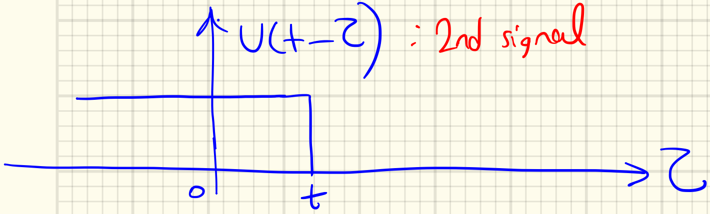
$$= \left(\int_0^t 1 \, d\tau \right) \underbrace{u(t)}_{t \geq 0} = \boxed{t \cdot u(t)}.$$

Graphical Method: ex: $u(t) * u(t) = \int_{-\infty}^{\infty} \underbrace{u(\tau)}.\underbrace{u(t-\tau)}\, d\tau = ?$

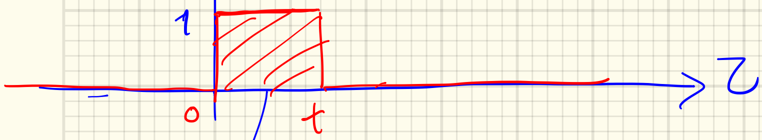
$u(\tau)$: 1st signal



$u(t-\tau)$: 2nd signal

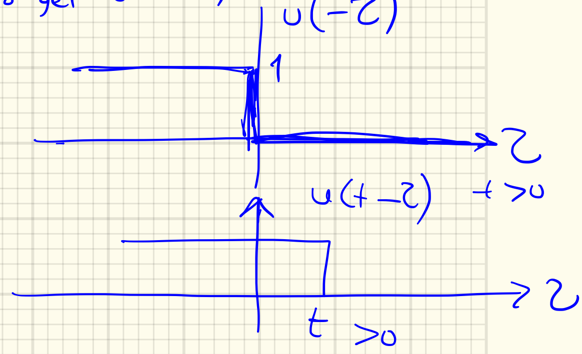


$u(\tau)u(t-\tau)$: pointwise multiply them

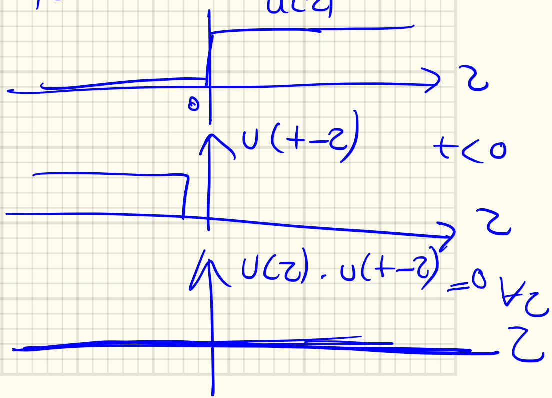


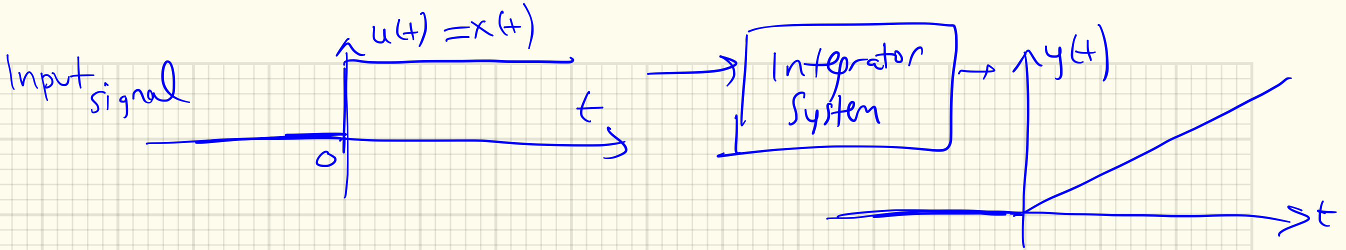
$$\int_0^t 1 \,.\, d\tau = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases} = t.u(t)$$

to get $u(t-\tau)$



for $t<0$

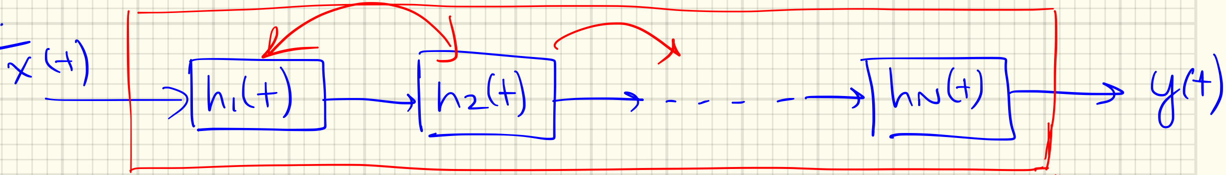




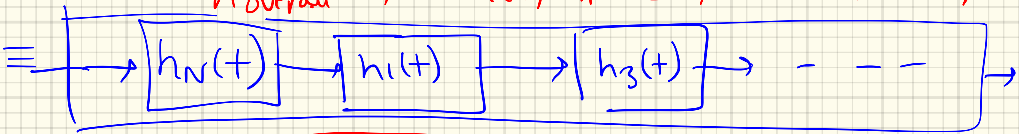
CASCADE & PARALLEL Connections for CT systems:

Cascade:

"in series" connection

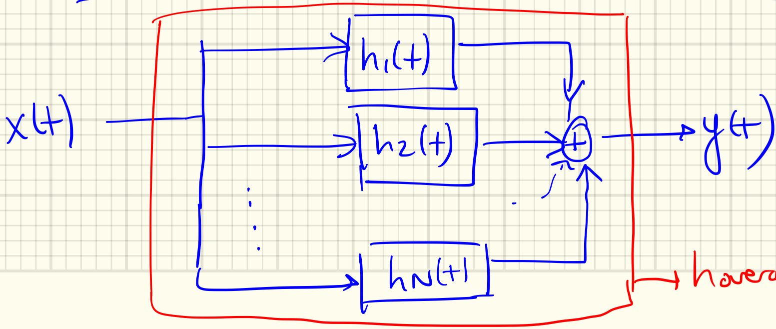


$$h_{overall}(t) = h_1(t) * h_2(t) * \cdots * h_N(t)$$



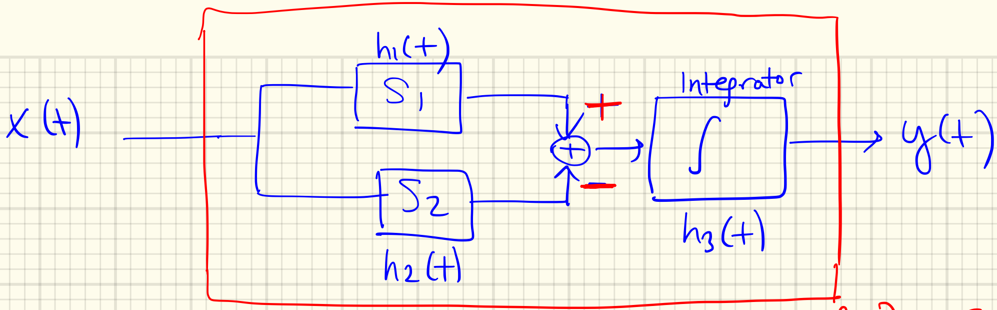
Due to commutativity & associativity

Parallel Connection of systems



$$\rightarrow h_{overall}(t) = h_1(t) + \cdots + h_N(t)$$

Ex:



$h_0(t) = ?$ Causal ? Stable ?

Given: $h_1(t) = \delta(t+1)$

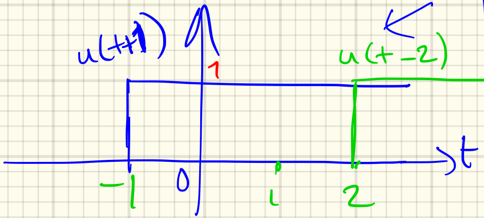
$h_2(t) = \delta(t-2)$

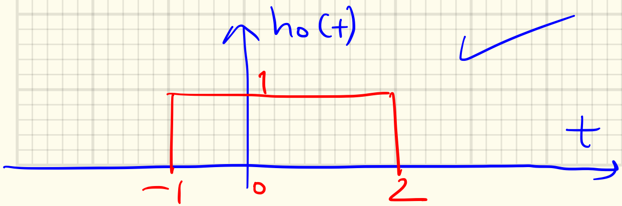
$h_3(t) = u(t)$

$h_0(t) = \left(h_1(t) - h_2(t)\right) * h_3(t)$

$h_0(t) = \left(\delta(t+1) - \delta(t-2)\right) * u(t)$ conv. distributive over addition

$h_0(t) = u(t+1) - u(t-2)$





LTI:

Stable system? $\displaystyle\int_{-\infty}^{\infty} |h(t)|\,dt \overset{?}{<} \infty$

$\displaystyle\int_{-1}^{2} 1 \cdot dt = 3.$ Stable.

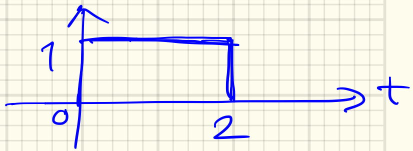
Causal ? LTI: $h(t) = 0$, $t < 0$?

Not causal.

Ex: $x(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t)$ (CT)

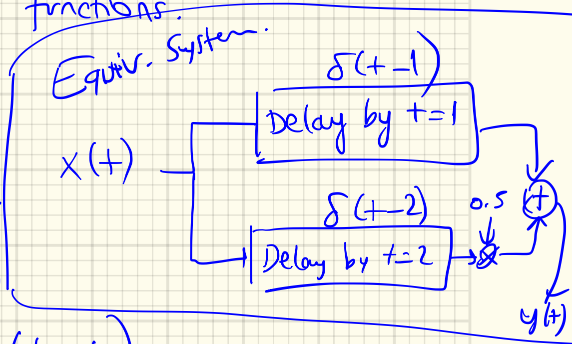
Given $h(t) = \delta(t-1) + 0.5\,\delta(t-2)$

Given $x(t)$



Q: Find $y(t)$.

Impulse response is given i.t.o δ functions.



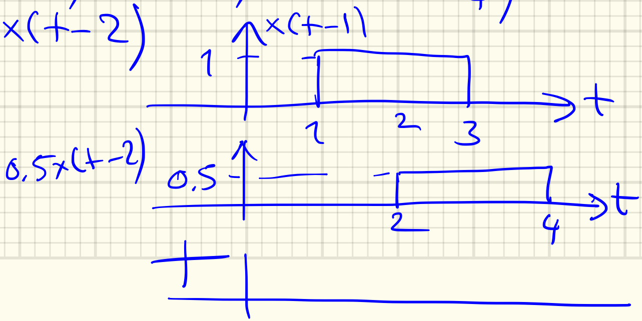
A. $x(t) = u(t) - u(t-2)$: I can write $x(t)$ in this way.

$$y(t) = x(t) * h(t) = x(t) * \left(\delta(t-1) + 0.5\,\delta(t-2)\right)$$

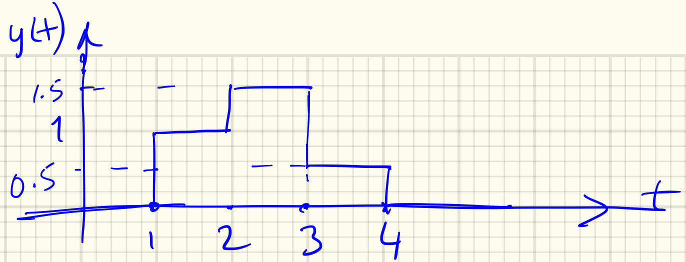
$$= \left(u(t) - u(t-2)\right) * \left(\delta(t-1) + 0.5\,\delta(t-2)\right)$$

$$y(t) = u(t-1) + 0.5\,u(t-2) - u(t-3) - 0.5\,u(t-4)$$

$$= \; y(t) = x(t-1) + 0.5\,x(t-2)$$

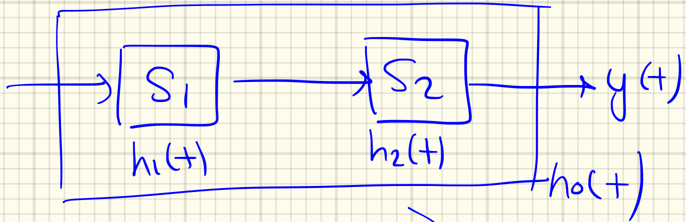


$\Rightarrow$

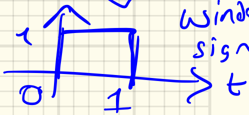


exercise:

Homework



Given $h_1(t) = \begin{cases} e^{-2t}, & 0 \le t \le 1 \\ 0, & o/w \end{cases}$ $\quad h_1(t) = e^{-2t} \cdot (u(t) - u(t-1))$



$h_2(t) = \delta^{(1)}(t)$ (differentiator) : $\left(y(t) = \frac{d}{dt} x(t) \right)$

Q: Is overall system $h_o(t)$ causal & stable?

Find & Plot $h_o(t)$.