BLG 354E Signals & Systems

26.04.2021

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FREQUENCY RESPONSE OF FIR FILTERS (DT)

(Chap 6 SP First ← DSP First)

$$x[n] \longrightarrow \boxed{\text{LTI } S} \longrightarrow y[n] = x[n] * h[n]$$

$h[n]$: impulse response

e.g. → FIR filter : $h[k] = \{ \overset{k=0}{\frac{1}{3}}, \frac{1}{3}, \frac{1}{3} \}$

$$y(n) = \sum_{k=0}^{2} h[k]\, x(n-k]$$

Frequency Response: how system responds to sinusoids of different frequencie

Let $x[n]$ be a sinusoidal input $\longrightarrow y(n) = ?$
(real/complex)

Recall DT complex exponential : $x[n] = Ae^{j\phi} e^{j\hat{\omega} n} = A e^{j(\hat{\omega} n + \phi)}$

$$\rightarrow y[n] = \sum_{k=0}^{M} \underbrace{h[k]}_{b_k} x[n-k] = \sum_{k=0}^{M} h[k]\, e^{j\hat{\omega}(n-k)} A e^{j\phi}$$

$$= \left(\sum_{k=0}^{M} h[k]\, e^{-j\hat{\omega} k} \right) \cdot \underbrace{Ae^{j\phi} e^{j\hat{\omega} n}}_{x[n]} = H(e^{j\hat{\omega}}) \cdot x[n]$$

$\triangleq H(e^{j\hat{\omega}})$: Frequency Response of the LTI system characterized by $h[n]$.

$$\underbrace{H(e^{j\hat{\omega}})}_{\text{a complex fn.}} = \underbrace{|H(e^{j\hat{\omega}})|} \cdot e^{j\underbrace{\angle H(e^{j\hat{\omega}})}}$$

$\rightarrow$ Phase Response of the system.

Frequency Response = Magnitude Response (Gain of the System) $\cdot\ e^{j\,\text{Phase Response of the system}}$

$$e^{j\hat{\omega}_0 n} \longrightarrow \underset{H(e^{j\hat{\omega}})}{\boxed{S}} \longrightarrow |H(e^{j\hat{\omega}_0})| \cdot e^{j\hat{\omega}_0 n} \cdot e^{j\angle H(e^{j\hat{\omega}_0})}$$

evaluated at input frequency

evaluated at input freq.

$$= H(e^{j\hat{\omega}_0}) \cdot \underbrace{e^{j\hat{\omega}_0 n}}_{x[n]}.$$

$\rightarrow$ Due Linearity of S:

$\rightarrow$ superposition applies

$$x[n] = \underbrace{\alpha_1}_{\text{real/complex}} e^{j\hat{\omega}_1 n} + \underbrace{\alpha_2}\, e^{j\hat{\omega}_2 n}$$

$\rightarrow$ $x(n)$ has 2 frequencies

$$\rightarrow y(n) = \alpha_1 H(e^{j\hat{\omega}_1}) \cdot \underbrace{e^{j\hat{\omega}_1 n}} + \alpha_2 H(e^{j\hat{\omega}_2}) \cdot \underbrace{e^{j\hat{\omega}_2 n}}$$

(Time Domain)

Frequency Response $\longleftrightarrow$ Impulse Response

$H(e^{j\hat{\omega}}) \longleftrightarrow h[n]$

Both characterize the system.

$$H(e^{j\hat{\omega}}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\hat{\omega}k}$$

: (DTFT) Discrete-Time Fourier Transform of the impulse response of the system.

Properties of $H(e^{j\hat{\omega}})$:

1) $H(e^{j\hat{\omega}})$ is periodic w/ 2π. $\left(\sum_k h[k]\, \underbrace{e^{-j(\hat{\omega} \mp 2\pi)k}}_{e^{-j\hat{\omega}k}}\right)$ ✓

real filter:

2) $\boxed{\text{When } h \text{ is real}}$; $H(e^{-j\hat{\omega}}) = H^*(e^{j\hat{\omega}})$

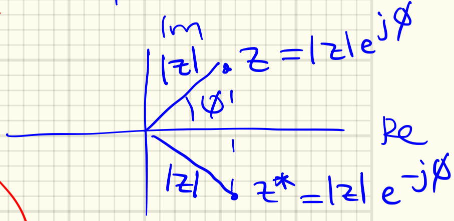
Apply the defn : $H(e^{j\hat{\omega}}) = \sum_k h[k]\, e^{-j\hat{\omega}k}$

take complex conjugate $\Big\{$ $H^*(e^{j\hat{\omega}}) = \left(\sum_k h[k]e^{-j\hat{\omega}k}\right)^* = \sum_k \underbrace{h^*[k]}_{=h[k]}\, \underbrace{e^{j\hat{\omega}k}}_{e^{-j(-\hat{\omega})k}}$

$= H(e^{-j\hat{\omega}k})$ →

$$H^*(e^{j\hat{\omega}}) = H(e^{-j\hat{\omega}})$$

for h real



$$|H(e^{j\hat{\omega}})|\, e^{-j\measuredangle H(e^{j\hat{\omega}})} = |H(e^{-j\hat{\omega}})|\, e^{j\measuredangle H(e^{-j\hat{\omega}})}$$

$$|H(e^{j\hat{\omega}})| = |H(e^{-j\hat{\omega}})| \Longrightarrow$$ Magnitude Response has EVEN symmetry $\rightarrow f(-x) = f(x)$

$$\underbrace{\measuredangle H(e^{-j\hat{\omega}})} = -\measuredangle H(e^{-j\hat{\omega}})$$

= angle fn. of $H(e^{j\hat{\omega}})$

= phase response

Phase Response has ODD SYMMETRY.

$f(-x) = -f(x)$

* Also note that $H(e^{j\hat{\omega}})$ is a function of only frequency,

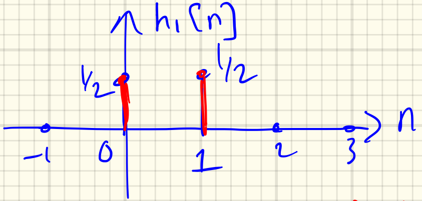
not time !!!

Ex:

$x[n] \longrightarrow \boxed{S_1} \longrightarrow y_1[n]$

I/O: $y_1[n] = \frac{1}{2}x[n] + \frac{1}{2}x[n-1]$

impulse response $\Rightarrow h_1[n] = \frac{1}{2}\delta[n] + \frac{1}{2}\delta[n-1]$



This system takes average of two consecutive values (current & previous)

→ Frequency Response

$$H_1(e^{j\hat{\omega}}) = \sum_{k=0}^{1} h(k)\, e^{-j\hat{\omega}k} = \underbrace{\frac{1}{2} + \frac{1}{2}e^{-j\hat{\omega}}}$$

$$= e^{-j\hat{\omega}/2}\underbrace{\left(\frac{e^{j\hat{\omega}/2} + e^{-j\hat{\omega}/2}}{2}\right)}$$

$$= e^{-j\hat{\omega}/2}\cos(\hat{\omega}/2)$$

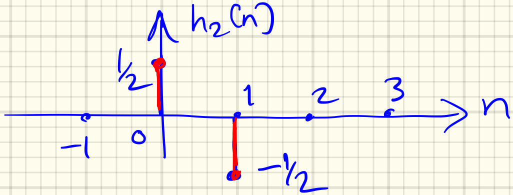
Smoothing filter: smooths the signal

2-pt RAF: Running Average Filter

$x[n] \longrightarrow \boxed{S_2} \longrightarrow y_2[n]$

I/O: $y_2[n] = \frac{1}{2}x[n] - \frac{1}{2}x[n-1]$

$\Rightarrow h_2[n] = \frac{1}{2}\delta[n] - \frac{1}{2}\delta[n-1]$



This system forms a difference btw two consecutive input values. (current & previous values)

$$H_2(e^{j\hat{\omega}}) = \frac{1}{2} - \frac{1}{2}e^{-j\hat{\omega}}$$

$$= j\, e^{-j\hat{\omega}/2}\left(\frac{e^{j\hat{\omega}/2} - e^{-j\hat{\omega}/2}}{2j}\right)$$

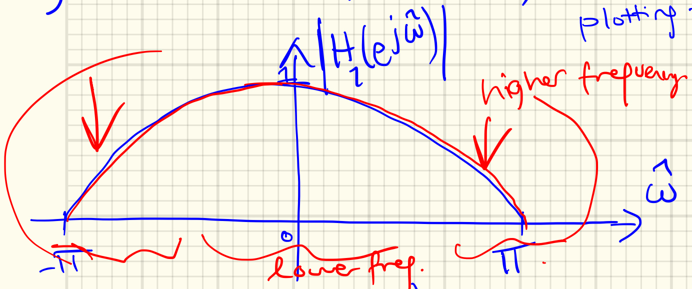
$$= e^{+j\pi/2}\, e^{-j\hat{\omega}/2} \cdot \sin(\hat{\omega}/2)$$

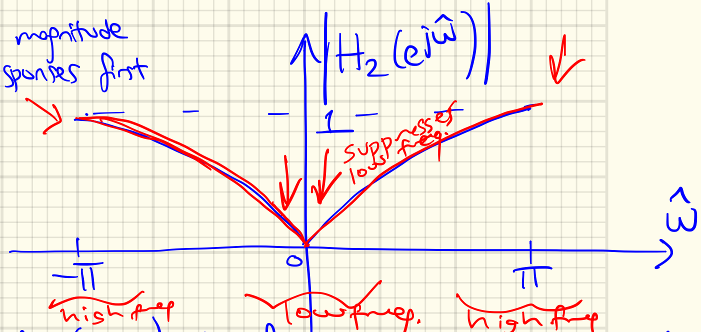
difference filter: tries to keep abrupt changes in the signal.

$H_1(e^{j\hat{\omega}}) = e^{-j\hat{\omega}/2} \cos\left(\frac{\hat{\omega}}{2}\right)$ $\quad$ $H_2(e^{j\hat{\omega}}) = e^{j(-\hat{\omega}/2 + \pi/2)} \cdot \sin\left(\frac{\hat{\omega}}{2}\right)$

↳ Plot these $\rightarrow |H(e^{j\hat{\omega}})| \cdot e^{j\angle H(e^{j\hat{\omega}})}$

plotting the magnitude responses first





(Recall w/ $\hat{\omega}$ (digital freq) we need to plot only $(-\pi, \pi)$ interval $\rightarrow$ b/c these are periodic w/ 2π

$$\underbrace{\sum_k e^{j\hat{\omega}_k n}}_{x(n)} \rightarrow \boxed{S_1} \rightarrow \underbrace{|H_1(e^{j\hat{\omega}})|}\ \sum_k e^{j\hat{\omega}_k n} \cdot e^{j\angle H_1(e^{j\hat{\omega}})}$$

H_2: $\boxed{\text{HIGH PASS FILTER}}$

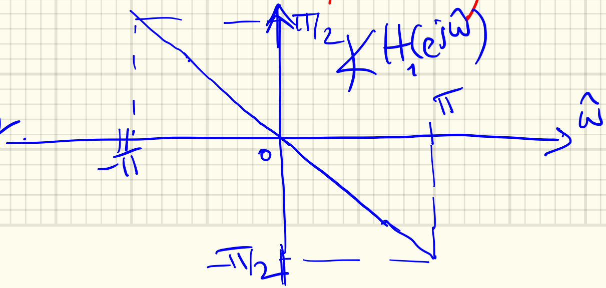
H_2: low frequencies are attenuated.

esp. at 0 freq: the gain $= 0$.

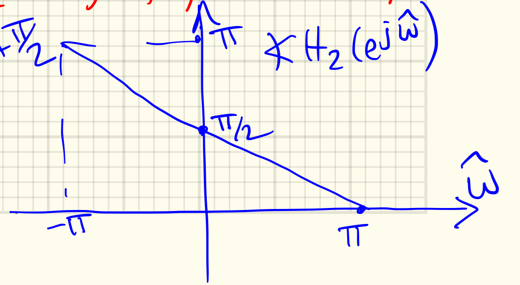
H_2: high frequencies are preserved

$H_1 \rightarrow$ preserves low frequencies, attenuates high frequencies $\Big\}$ H_1 $\boxed{\text{LOWPASS Filter}}$

Phase Response $\angle H_1 = -\hat{\omega}/2$

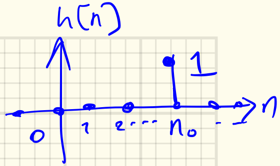


$\angle = -\frac{\hat{\omega}}{2} + \pi/2$



Ex: $x(n) \longrightarrow \boxed{S} \longrightarrow y(n) = x[n-n_0]$

Impulse Response: $h[n] = \delta[n-n_0]$ Delay (by n_0) System

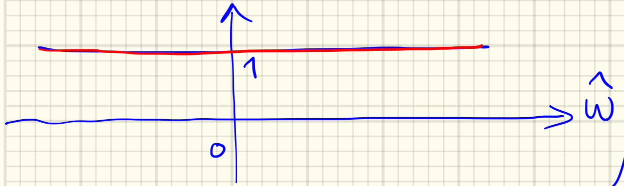


$$H(e^{j\hat{\omega}}) = \sum_{k=-\infty}^{\infty} \underbrace{h[k]}\, e^{-j\hat{\omega}k} = e^{-j\hat{\omega}n_0}$$

$$= \sum_{k} \overbrace{\delta[k-n_0]}^{=1 \text{ at } k=n_0} e^{-j\hat{\omega}k}$$

$$H(e^{j\hat{\omega}}) = e^{-j\hat{\omega}n_0}$$

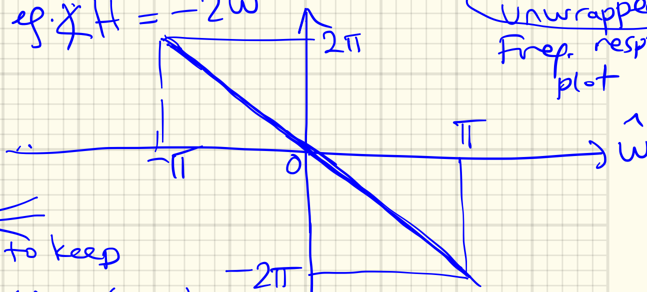
$$|H(e^{j\hat{\omega}})| = \underline{1}$$



$$\measuredangle H(e^{j\hat{\omega}}) = -\hat{\omega}n_0$$

Unwrapped Freq. response plot

e.g. $\measuredangle H = -2\hat{\omega}$

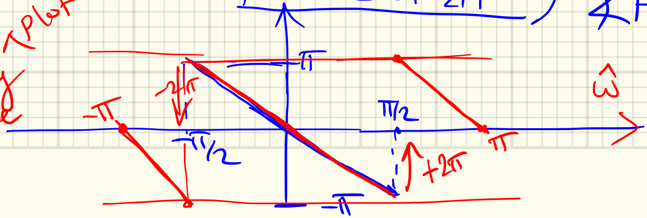


want to keep $\measuredangle H(\cdot) \in (-\pi,\pi)$ range

Re plot

Recall $\hat{\omega}$ is unique in $(-\pi,\pi)$
periodic w/ 2π

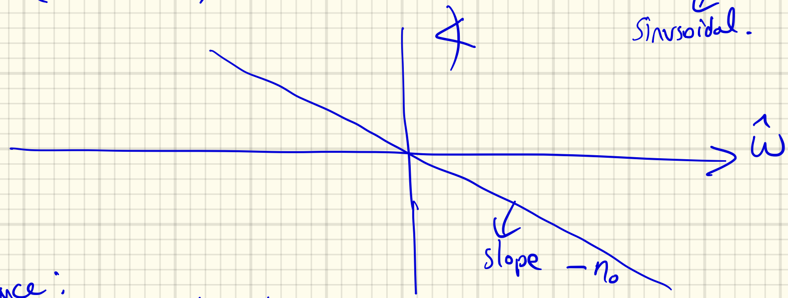
Wrapped Frequency Response plot

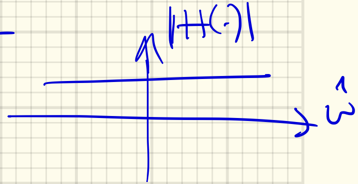


→ **Linear Phase Filter**: introduce no distortion (Gain $=1$ $\forall \hat{\omega}$), other than time delay

Note: This is a desired characteristic.

$$\underset{\text{(sinusoidal)}}{x(n)} \longrightarrow \boxed{\begin{matrix}\text{Linear Phase}\\ H(e^{j\hat{\omega}})\end{matrix}} \longrightarrow \underbrace{x(n)}_{\text{sinusoidal.}}\ \underbrace{H(e^{j\hat{\omega}})}_{\underbrace{|H(\cdot)|}_{=1}} \cdot \underbrace{e^{-j\hat{\omega} n_0}}$$

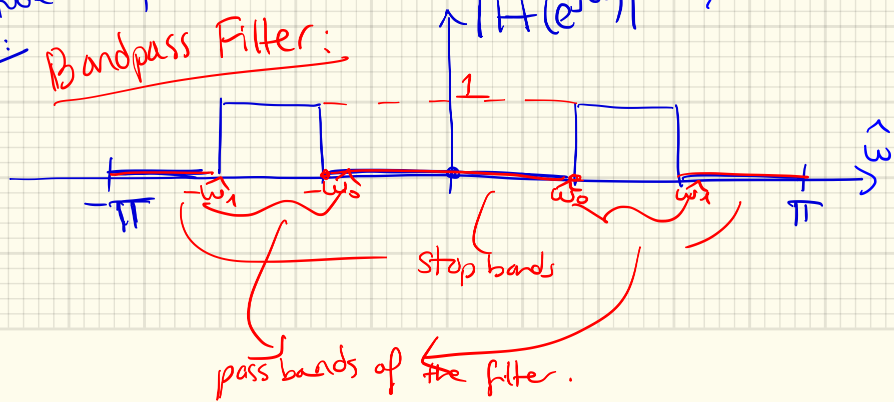




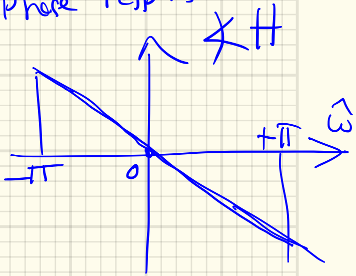
For instance:

Magnitude Response has this characteristic, $\measuredangle$

Ideal Filter: Bandpass Filter:

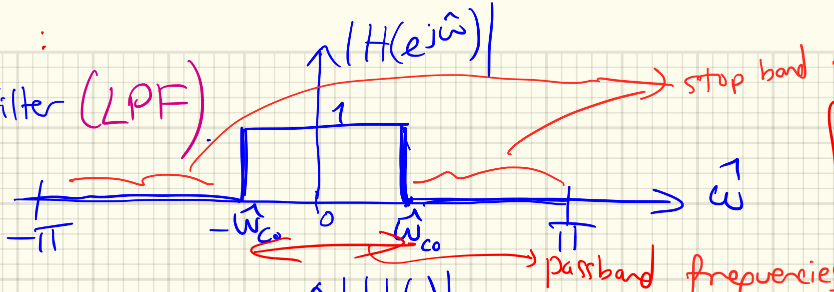


Phase response



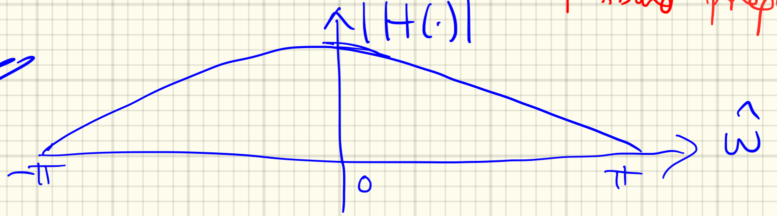
IDEAL FILTERS :

Low Pass : Ideal Filter (LPF)



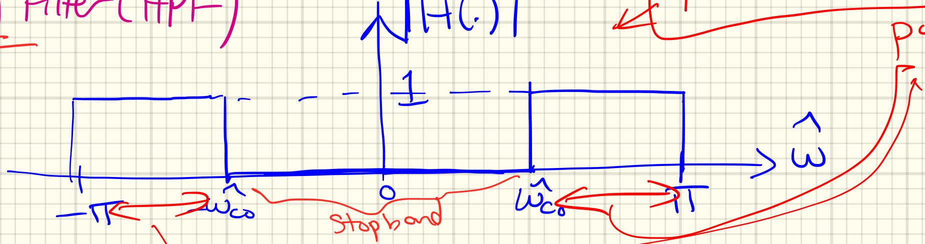
Note the even symmetry in these filters

Recall 2pt RAF : Non-ideal : LPF

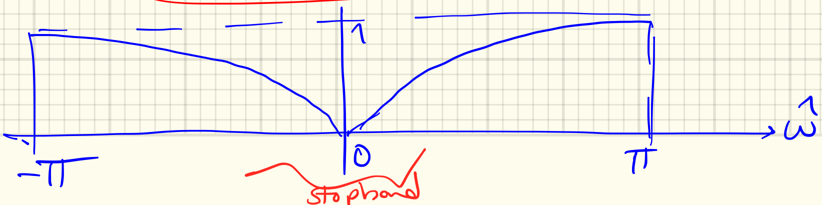


parameter $\hat{\omega}_{co}$: cut-off frequency

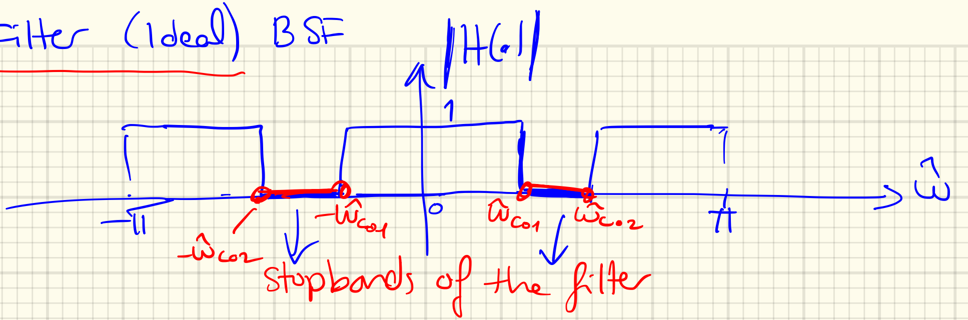
High Pass (Ideal) Filter (HPF)



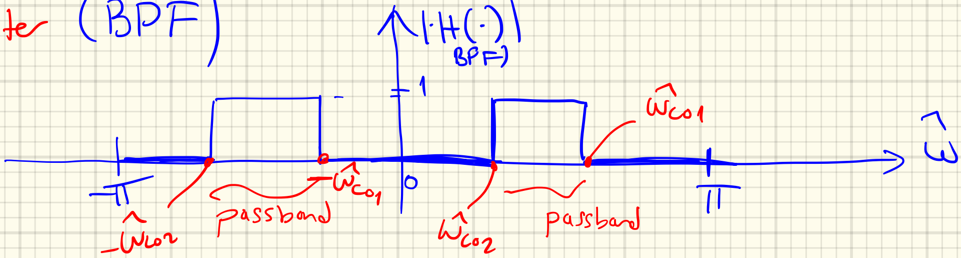
Recall 1st order difference FIR Filter : Non-ideal HPF



Bandstop Filter (Ideal) BSF



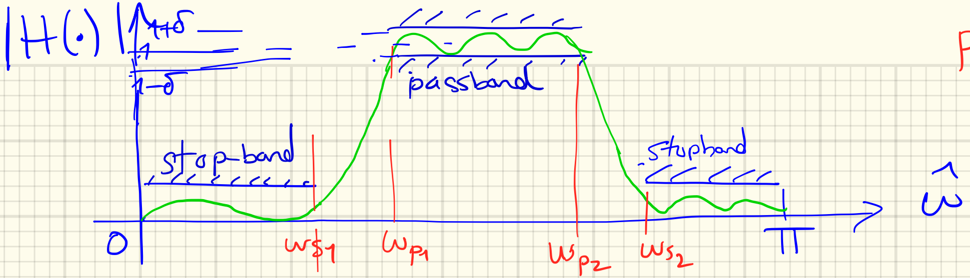
Bandpass Filter (BPF)



Note: Ideal filters are good for mathematical abstraction. but cannot be implemented.

Practical Filters $\longrightarrow$ real filters $\rightarrow$ symmetric

so let's look at them btw $(0, \pi)$ b/c $(-\pi, 0)$ is its reflection.



Practical Filter

Ideal Filter

Note: DTFT: $y[n] = x[n] * h[n]$

$$Y(e^{j\hat{\omega}}) = \sum_n \underbrace{y[n]}_{\sum_k h(k)x[n-k]} e^{-j\hat{\omega}n} = \sum_n \sum_k h(k]\, x[n-k]\, e^{-j\hat{\omega}n}$$

let $m = n-k \rightarrow n = m+k$

$$= \sum_k \sum_m h[k)\, x[m]\, e^{-j\hat{\omega}(m+k)}$$

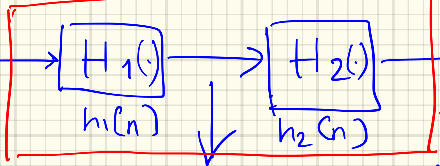
$$= \underbrace{\sum_k h(k)\, e^{-j\hat{\omega}k}}_{H(e^{j\hat{\omega}})} \underbrace{\sum_m x(m]\, e^{-j\hat{\omega}m}}_{X(e^{j\hat{\omega}})}$$

CONVOLUTION in time domain $\Longleftrightarrow$ MULTIPLICATION in Frequency domain

$$\underbrace{e^{j\hat{\omega}_0 n}}$$

CASCADED LTI Systems:

$x[n] = e^{j\hat{\omega} n}$



$\rightarrow y[n] = ?$

$$H_{overall}(e^{j\hat{\omega}}) = H_1(e^{j\hat{\omega}}) \cdot H_2(e^{j\hat{\omega}})$$

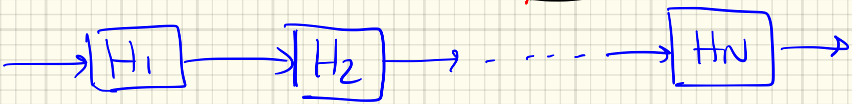
$H_1(e^{j\hat{\omega}}) \cdot e^{j\hat{\omega} n}$

$$H_2(e^{j\hat{\omega}}) \cdot H_1(e^{j\hat{\omega}}) \cdot e^{j\hat{\omega} n}$$

pointwise multiply

Recall: in time domain this was convolution operation

$e^{j\hat{\omega} n}$



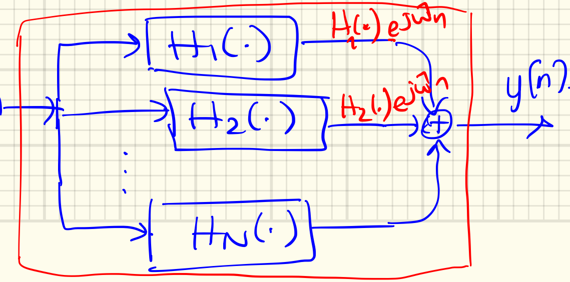
$\rightarrow y[n] = ?$

★ We can change the order of cascaded LTI system due to commutativity of the multiplication.

$$y[n] = H_1(e^{j\hat{\omega}}) \cdots H_N(e^{j\hat{\omega}}) e^{j\hat{\omega} n} = \prod_{i=1}^{N} H_i(e^{j\hat{\omega}}) \cdot e^{j\hat{\omega} n}$$

PARALLEL LTI Systems:

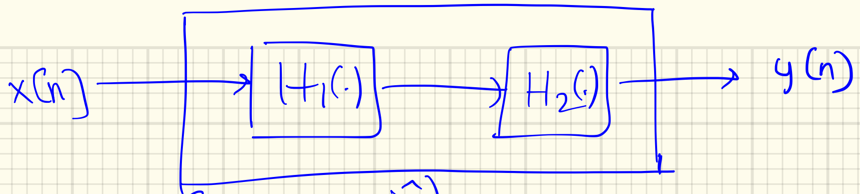
$x[n] = e^{j\hat{\omega} n}$



$$y[n] = \underbrace{\left[H_1(e^{j\hat{\omega}}) + H_2(e^{j\hat{\omega}})\right]}_{} e^{j\hat{\omega} n}$$

$$H_{overall}(e^{j\hat{\omega}}) = \sum_{i=1}^{N} H_i(e^{j\hat{\omega}})$$

Ex).



Note: Know going from
$h[n] \to H(e^{j\hat{\omega}})$
$H(e^{j\hat{\omega}}) \to h[n]$

Q. Find $h_3[n]$ using Freq. Response.

Q: $H_3(e^{j\hat{\omega}}) = ? \longrightarrow h_3[n] = ?$

Given

$$h_1[n] = 2\delta[n] + 4\delta[n-1] + 4\delta[n-2] + 2\delta[n-3]$$

$$h_2[n] = \delta[n] - 2\delta[n-1] + \delta[n-2]$$

Soln:

$$H_3(e^{j\hat{\omega}}) = \underbrace{H_1(e^{j\hat{\omega}})} \cdot H_2(e^{j\hat{\omega}})$$

$$H_1(e^{j\hat{\omega}}) = \sum_k h_1[k] e^{-j\hat{\omega}k} = 2 + 4e^{-j\hat{\omega}} + 4e^{-j\hat{\omega}2} + 2e^{-j\hat{\omega}3}$$

$$H_2(e^{j\hat{\omega}}) = \sum_k h_2[k] e^{-j\hat{\omega}k} = 1 - 2e^{-j\hat{\omega}} + e^{-j\hat{\omega}2}$$

$$H_3(e^{j\hat{\omega}}) = \left(2 + 4e^{-j\hat{\omega}} + 4e^{-j2\hat{\omega}} + 2e^{-j3\hat{\omega}}\right)\left(1 - 2e^{-j\hat{\omega}} + e^{-j2\hat{\omega}}\right)$$

$$\vdots$$

$$H_3(e^{j\hat{\omega}}) = 2 - 2e^{-j\hat{\omega}2} - 2e^{-j\hat{\omega}3} + 2e^{-j\hat{\omega}5}$$

$$\hookrightarrow h_3[n] = 2\delta[n] - 2\delta[n-2] - 2\delta[n-3] + 2\delta[n-5]$$

→ Sum of Sinusoids $\longrightarrow \underset{\text{LTI}}{\boxed{H(e^{j\hat{\omega}})}} \longrightarrow \sum H(e^{j\hat{\omega}}) \cdot$ sinusoids

b/c superposition applies to LTI systems.

Set $x[n] = A_0 + A_1 \cos(\hat{\omega}_1 n + \phi) + (A_2 \cos(\hat{\omega}_2 n) + - -)$ can be added later

$$x(n) = A_0 + \frac{A_1}{2} e^{j\hat{\omega}_1 n} \cdot e^{j\phi} + \frac{A_1}{2} e^{-j\hat{\omega}_1 n} \cdot e^{-j\phi}$$

$$e^{j\hat{\omega}_1 n} \longrightarrow \boxed{H(\cdot)} \longrightarrow H(e^{j\hat{\omega}_1}) \cdot e^{j\hat{\omega}_1 n}$$

$$e^{j(-\hat{\omega}_1)n} \longrightarrow \boxed{H(\cdot)} \longrightarrow H(e^{j(-\hat{\omega}_1)})\, e^{-j\hat{\omega}_1 n}$$

$$A_0 \cdot e^{j0} \longrightarrow \boxed{H(\cdot)} \longrightarrow H(e^{j0}) \cdot A_0$$

$$\rightarrow y[n] = H(e^{j0}) A_0 + \frac{A_1}{2} \underbrace{H(e^{j\hat{\omega}_1})}_{|H(e^{j\hat{\omega}_1})|\, e^{j\measuredangle H(e^{j\hat{\omega}_1})}} e^{j\hat{\omega}_1 n} \cdot e^{j\phi} + \frac{A_1}{2} \underbrace{H(e^{-j\hat{\omega}_1})}_{\underbrace{|H(e^{-j\hat{\omega}_1})|}_{=|H(e^{j\hat{\omega}_1})|}\, e^{j\underbrace{\measuredangle H(e^{-j\hat{\omega}_1})}_{-\measuredangle H(e^{+j\hat{\omega}_1})}}} e^{-j\hat{\omega}_1 n} \cdot e^{-j\phi}$$

$$y(n) = H(e^{j0}) A_0 + A_1 \left| H(e^{j\hat{\omega}_1}) \right| \left(\frac{e^{j\hat{\omega}_1 n} e^{j\phi} \cdot e^{j\measuredangle H(e^{j\hat{\omega}_1})} + e^{-j\hat{\omega}_1 n} e^{-j\phi} \cdot e^{-j\measuredangle H(e^{j\hat{\omega}_1})}}{2} \right)$$

→ phase part.

$$y[n] = H(e^{j0})\,A_0 + \left|H(e^{j\hat{\omega}_1})\right| \cdot A_1 \cos\left(\hat{\omega}_1 n + \phi + \angle H(e^{j\hat{\omega}_1})\right)$$

$$A_0 + A_1\cos(\hat{\omega}_1 n + \phi) \longrightarrow \boxed{H(\cdot)} \longrightarrow \begin{array}{l} A_0\, H(e^{j0}) \\ + \\ \left|H(\cdot)\right|_{\hat{\omega}=\hat{\omega}_1} A_1 \cdot \underbrace{\cos(\hat{\omega}_1 n + \phi}_{\text{original cosine}} + \underbrace{\angle H(\cdot)}_{\text{evaluated at } \hat{\omega}_1}) \end{array}$$

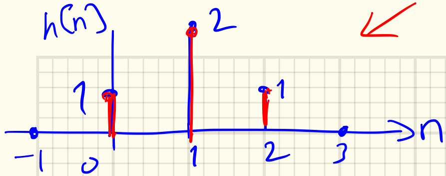
cosine input is modulated by the gain $|H(\cdot)|\big|_{\omega=\hat{\omega}_1}$ of the system, & its phase is shifted by $\angle H(\cdot)\big|_{\hat{\omega}=\hat{\omega}_1}$.

Ex: For FIR filter: $h[n] = \{\underset{\uparrow n=0}{1}, 2, 1\}$

Find response to $x[n] = 4 + 3\cos\left(\frac{\pi}{3}n - \frac{\pi}{2}\right) + 3\cos\left(\frac{7\pi}{8}n\right) + \delta[n-2]$

Input

1) Find Frequency Response of the system:



$$H(e^{j\hat\omega}) = ?$$

$$= \sum_{k=0}^{2} h[k] e^{-j\hat\omega k}$$

$$H(e^{j\hat\omega}) = 1 + 2e^{-j\hat\omega} + e^{-j\hat\omega 2}$$

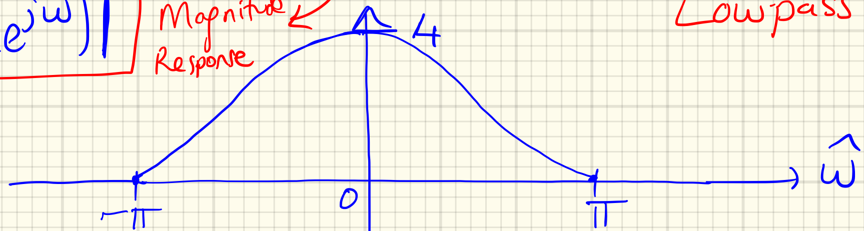
$$= e^{-j\hat\omega}\left(e^{j\hat\omega} + 2 + e^{-j\hat\omega}\right) = e^{-j\hat\omega}\left(2 + 2\underbrace{\frac{e^{+j\hat\omega} + e^{-j\hat\omega}}{2}}_{\cos(\hat\omega)}\right)$$

phase response part

$$H(e^{j\hat\omega}) = \boxed{e^{-j\hat\omega}}\underbrace{\left(2 + 2\cos\hat\omega\right)}_{}$$

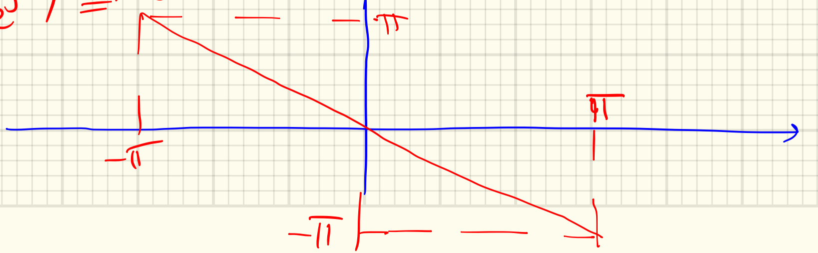
$$\boxed{|H(e^{j\hat\omega})|}$$

Magnitude Response



Lowpass Filter ✓

$$\angle H(e^{j\hat\omega}) = -\hat\omega$$



$\hat\omega$

Linear Phase ✓

exercise: we'll complete this example next time

continue w/ superposition rule.