3D Vision

BLG634E

Gözde ÜNAL

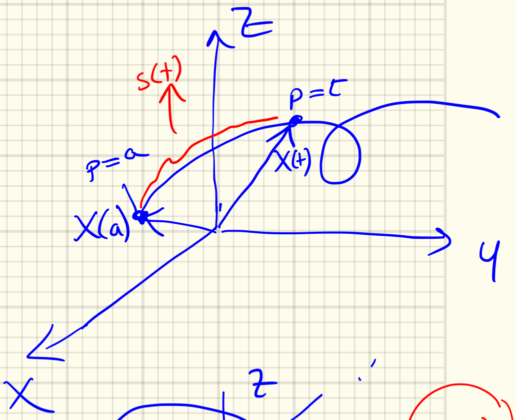
Differential Geometry
of Curves & Surfaces

01.06.2021
08.06.2021

01.06.2021

Arc length parameterization : vs Arbitrary parameterization → $\underline{X}(t) = (x(t), y(t), z(t))$

$$s(t) = \int_a^t |\underline{X}'(p)|\, dp \quad : \text{Arc length}$$



Using 1st fundamental thm of calculus:

$$\frac{ds}{dt} = |x'(t)|$$

Ex: reparameterize a helix : w.r.t. arc length:

$\underline{X}(t) = (\cos t, \sin t, t)$ $t \in (0, a)$

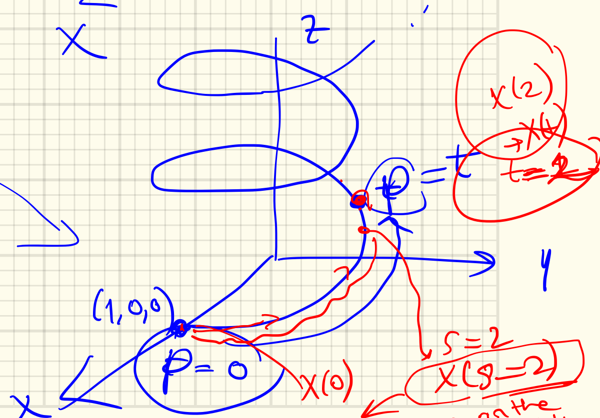
$\underline{x}'(t) = (-\sin t, \cos t, 1)$

$|x'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

$s(t) = \int_0^t |x'(p)|\, dp = \int_0^t \sqrt{2}\, dp = \sqrt{2}\, t \to t = \dfrac{s}{\sqrt{2}}$

reparameterize $\underline{x}(t(s)) = \left(\cos\left(\dfrac{s}{\sqrt{2}}\right), \sin\left(\dfrac{s}{\sqrt{2}}\right), \dfrac{s}{\sqrt{2}}\right)$

$\underline{x}(s) =$

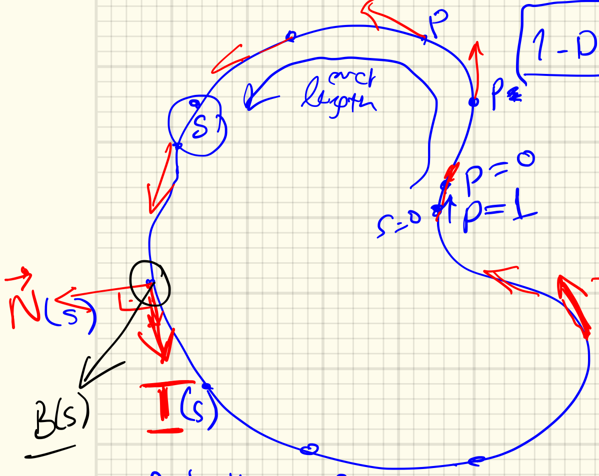


tells us where we are on the curve after we walk 2 units on the curve

(07.06.2021)

Last time: Curves : $\boxed{\text{1-D manifold}}$: $\underline{C}(p) = \begin{pmatrix} x(p) \\ y(p) \\ z(p) \end{pmatrix}$

$p \in [0,1] = I \subset \mathbb{R}$



$\underline{C} : I \to \mathbb{R}^3$: 3D space curve

$I \to \mathbb{R}^2$: 2D curve

$\underline{T} = \underline{C}'(p) = \frac{\partial \underline{c}}{dp}$: Tangent vector to the curve at a pt p.

Regular Curve : $\underline{C}'(p) \neq \underline{0}$ i.e. Tangent vector should be defined.

normal to the curve

$\underline{N}(s) = \underline{T}'(s)$

$\propto C''(s)$

$ds = \|C_p\| dp \leftarrow$ $\boxed{\underline{S}(p) = \int_0^p \|\underline{C}'(t)\| \, dt}$: arc length.

$\langle T, T \rangle = 1$

$\hookrightarrow \langle T', T \rangle = 0$

$\uparrow$

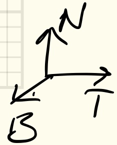
$\underline{N}$.

$B(s) = \underline{T}(s) \times \underline{N}(s)$

$\hookrightarrow$ relates to τ : torsion

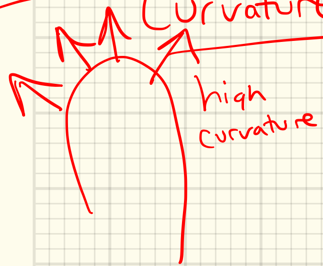
s: arclength param

Local Frenet frame: $T(s)$, $N(s)$, $B(s)$



$\vec{N}$; T'

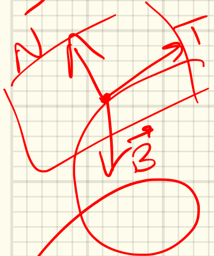
Curvature $\propto \|C''(s)\|$: how much the curve bends



$\vec{N} \propto T'(s) \propto C''(s)$

Curvature $K = \|C''(s)\|$

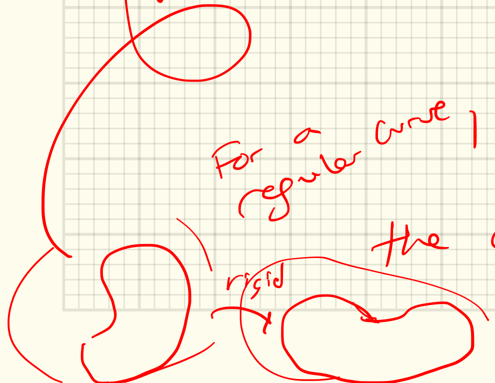
→ Torsion : $\tau = \|B'(s)\|$: how much the curve twists.



Fundamental Theorem of Local Theory of Curves

Do Carmo → Reference book

For a regular curve, Given $K(s)$, $\tau(s)$, → this defines the curve uniquely up to a rigid motion.



K, τ ✓

SE(3) ; SE(2).

R, T.

Differential Geometry of Surfaces (Do Carmo Book)

Like curves in 2D / 3D

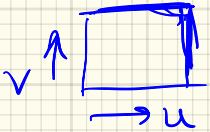
1D manifold — regular curve

$C: \underset{[0,1]}{I} \xrightarrow{\text{map}} \mathbb{R}^2, \mathbb{R}^3 \; : C(p) = \begin{pmatrix} x(p) \\ y(p) \end{pmatrix},$

$C'(p) \neq 0 \quad \forall p \in [0,1]$

X & Y are continuous functions on I.

For a surface: 2D manifold



A patch is a piece of a surface.

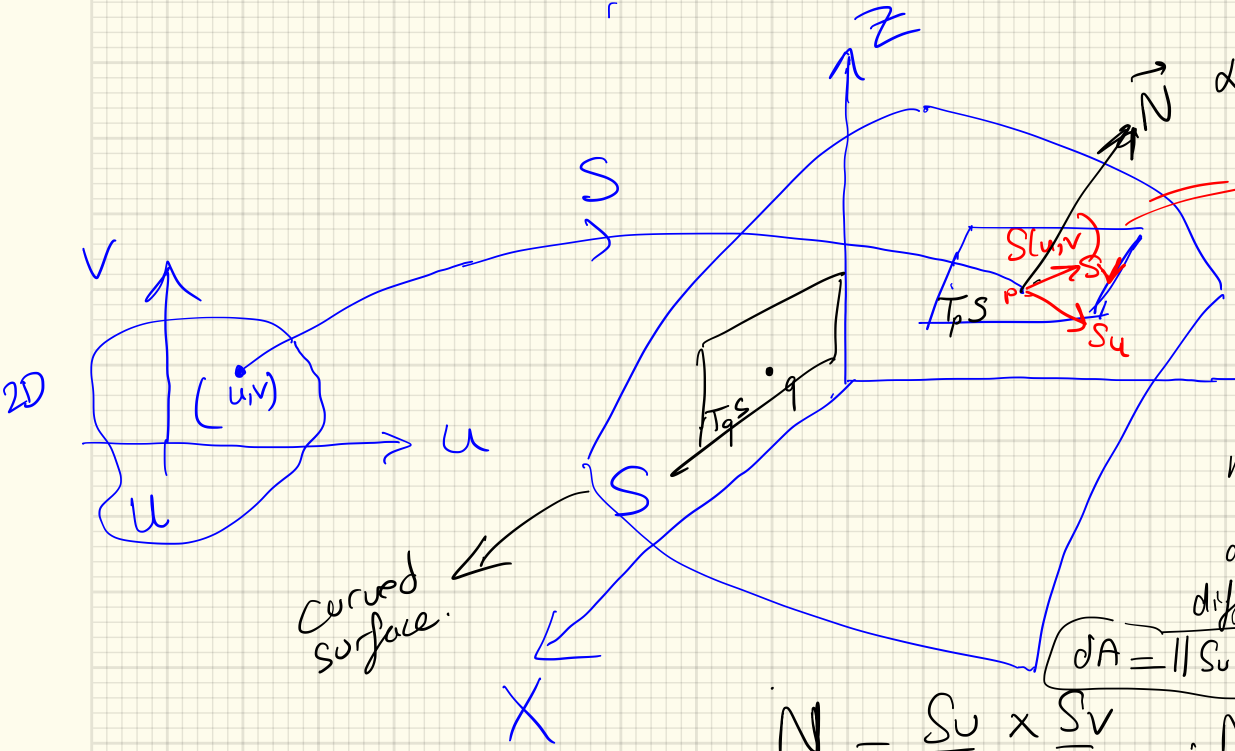
Def: A surface is a map:

$$S : \overset{I}{[0,1]} \times \overset{I}{[0,1]} \longrightarrow \mathbb{R}^3 :$$

$$S(u, v) \longrightarrow \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$

a parameterization of your surface S.

$S : U \to \mathbb{R}^3$



$\vec{N} \propto \underline{S_u} \times \underline{S_v}$

$S(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$

$\left.\begin{matrix} \underline{S_u} \\ \underline{S_v} \end{matrix}\right\}$ tangent vectors

Recall:

$ds = \|C_p\| dp.$

differential arc length

differential area:

$$\boxed{dA = \| S_u \times S_v \| \, du \, dv}$$

$$\underline{N}_p = \frac{\underline{S_u} \times \underline{S_v}}{\| \underline{S_u} \times \underline{S_v} \|}$$: Normal to the surface at point p

T_pS: tangent plane at point p on the surface S.

Def: Regular Surface S: $S(u,v)$ is differentiable

$$S(u,v) = \begin{pmatrix} X(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$

have continuous partial derivatives

&

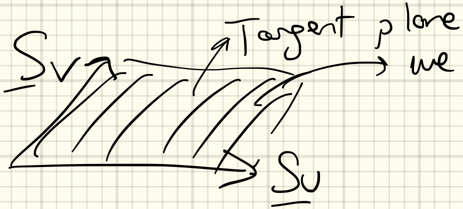
$$\underline{S}_u \neq 0 \qquad = \frac{\partial S}{\partial u}$$

$$\underline{S}_v \neq 0 \qquad = \frac{\partial S}{\partial v}$$

$$\underline{S}_u \times \underline{S}_v \neq 0$$

$\underline{S}_u$, $\underline{S}_v$ are tangent vectors on the tangent plane to the surface S at point p.

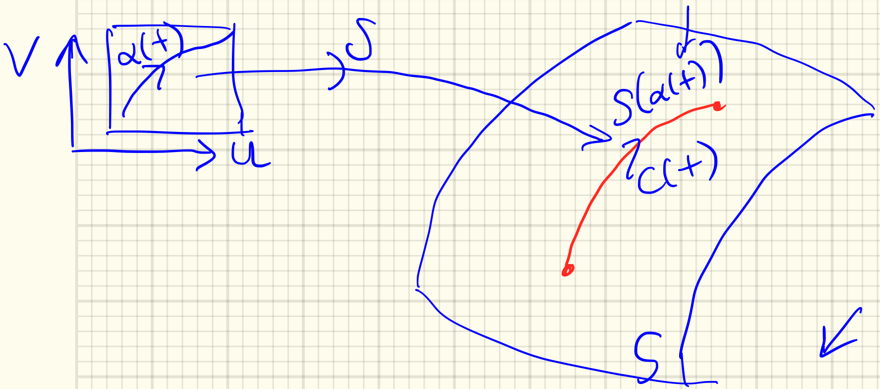
$a\underline{S}_u + b\underline{S}_v$ are tangent vectors on the tangent plane



we can have arbitrarily many tangent vectors on this plane.

Def: Tangent vectors $\underline{S_u}$, $\underline{S_v}$ form a vector space of dimension 2, called the Tangent Plane at p: $T_p(S)$: $\underline{S_u}(p)$, $\underline{S_v}(p)$ are basis vectors of $T_p(S)$.

The First Fundamental Form: (Riemannian Metric)



$C(t) = S(\alpha(t))$

curve on the surface

$C(t) = S(u(t), v(t))$

$C'(t) = \underline{S_u}\, u_t + \underline{S_v} \cdot v_t$

$\|C'(t)\|^2 = \langle \underline{C}'(t), C'(t) \rangle$

$$\|C'(t)\|^2 = \langle (\underline{S_u}\, u_t + \underline{S_v} \cdot v_t), (\underline{S_u}\, u_t + \underline{S_v}\, v_t) \rangle$$

inner product in $\mathbb{R}^3$

$$\rightarrow \|C'(t)\|^2 = \underbrace{(\underline{S}_u \cdot \underline{S}_u)}_{E} u_t^2 + 2\underbrace{(\underline{S}_u \cdot \underline{S}_v)}_{F} u_t v_t + \underbrace{(\underline{S}_v \cdot \underline{S}_v)}_{G} v_t^2$$

E, F, G : 1^{st} Fundamental form coefficients

$$\|C'(t)\|^2 = \begin{bmatrix} u_t & v_t \end{bmatrix} \underbrace{\begin{bmatrix} \underline{S}_u \cdot \underline{S}_u & \underline{S}_u \cdot \underline{S}_v \\ \underline{S}_u \cdot \underline{S}_v & \underline{S}_v \cdot \underline{S}_v \end{bmatrix}}_{} \begin{bmatrix} u_t \\ v_t \end{bmatrix}$$: quadratic form

$\underline{\underline{I}}_p$: 1^{st} fundamental form at p :

$\underline{\underline{I}}_p$: positive definite & symmetric matrix.

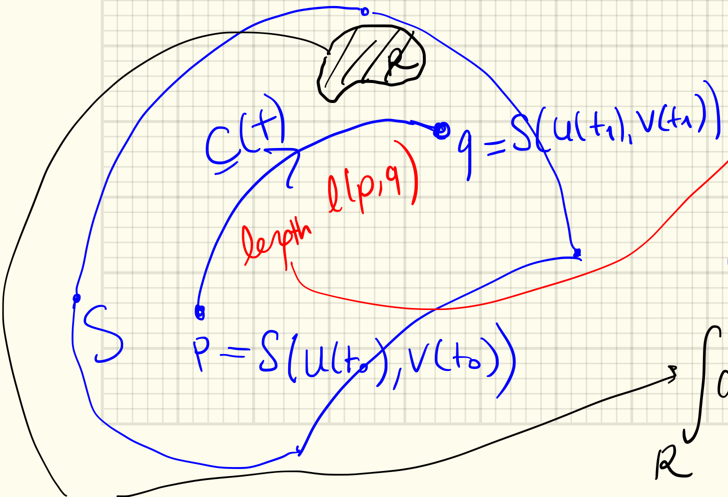
$$\underline{\underline{I}}_p = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$$

$\longrightarrow$

$$
\left.\begin{aligned}
E &= \underline{S}_u \cdot \underline{S}_u \\
F &= \underline{S}_u \cdot \underline{S}_v \\
G &= \underline{S}_v \cdot \underline{S}_v
\end{aligned}\right\}
$$

1st Fundamental Form $\underline{I}_p = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$

★ On a curved surface (contrary to a flat surfaces), the inner product induced by the "Riemannian metric" on the tangent space at every point changes as the point moves on the surface.

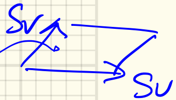


Calculate the length:

$$
l(p,q) = \int_{t_0}^{t_1} \sqrt{E\,U_t^2 + 2F\,U_t V_t + G\,V_t^2}\, dt
$$

Similarly to calculate area

$$
dA = \boxed{\| \underline{S}_u \times \underline{S}_v \|}\, du\, dv
$$



$$
\int_R dA = \int \sqrt{EG - F^2}\; du\, dv
$$

Note: Lagrange's identity:

$$\|\underline{a} \times \underline{b}\|^2 = \|a\|^2 \|b^2\| - (a \cdot b)^2$$

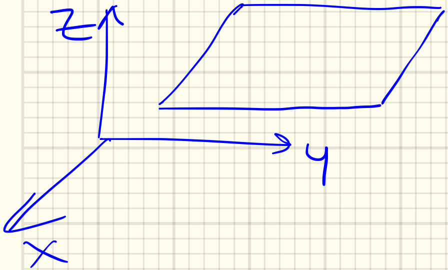
$$\|\underline{S_u} \times \underline{S_v}\| = (\underline{S_u} \cdot \underline{S_u})(\underline{S_v} \cdot \underline{S_v}) - (\underline{S_u} \cdot \underline{S_v})^2$$

$$= E \cdot G - F^2$$

Area of a bounded region R on the surface S:

$$A = \int_R \sqrt{EG - F^2}\, du\, dv$$

Ex: A plane:



$$S(u,v) = \begin{pmatrix} u \\ v \\ \text{const} \end{pmatrix} \in \mathbb{R}^3:$$

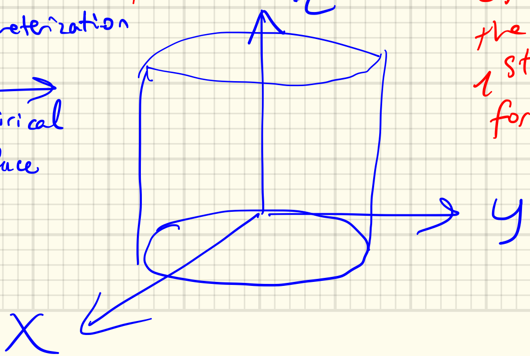
A plane parameterization

Q: Caculate the 1st Fund. Form coef. E, F, G?

$$S_u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad S_v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{aligned} E &= S_u \cdot S_u = 1 = E \\ F &= S_u \cdot S_v = 0 = F \\ G &= S_v \cdot S_v = 1 = G \end{aligned} \qquad \underline{\underline{I_p}} = \text{Identity } 2\times 2 \text{ Matrix.}$$

Ex: $S(u,v) = \begin{pmatrix} \cos u \\ \sin u \\ v \end{pmatrix}$ parameterization of a cylindirical surface



$$S_u = \begin{pmatrix} -\sin u \\ \cos u \\ 0 \end{pmatrix} \qquad S_v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\boxed{\begin{aligned} E &= 1 \\ F &= 0 \\ G &= 1 \end{aligned}}$$

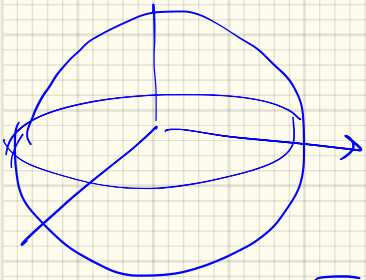
★ plane & Cylinder have the same 1st fund. form! $\Rightarrow$

→ 1st fundamental form does not characterize a surface, itself.

Ex: Sphere

$$S(\theta, \varphi) = \begin{pmatrix} \sin\theta \cos\varphi \\ \sin\theta \sin\varphi \\ \cos\varphi \end{pmatrix}$$

$\theta \in (0, \pi)$
$\varphi \in (0, 2\pi)$
$a, b, c = 1$



$S_\theta = \checkmark$

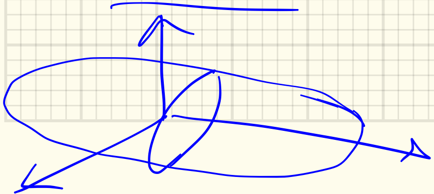
$S_\varphi = \checkmark$

$$\left.\begin{aligned} E &= 1 = S_\theta \cdot S_\theta \\ F &= 0 = S_\theta \cdot S_\varphi \\ G &= \sin^2\theta = S_\varphi \cdot S_\varphi \end{aligned}\right\} \underline{\underline{I}} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix}$$

Ex: Ellipsoid:

$$S(u, v) = \begin{pmatrix} a \sin u \cos v \\ b \sin u \sin v \\ c \cdot \cos u \end{pmatrix}$$

$0 < u < \pi$
$0 < v < 2\pi$
$a, b, c \neq 0$



$\Longrightarrow$

$\Rightarrow$ Implicit Surface representation : $S = \{ (x, y, z) \in \mathbb{R}^3 : f(x, y, z) = c \}$ constant in $\mathbb{R}$

eg. ellipsoid

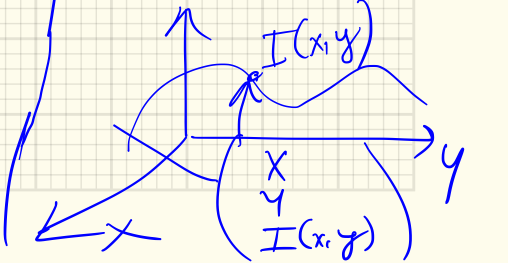
$$\boxed{\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1}$$

explicit parameterization : $S(u,v) = f^{-1}(c) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$

eg. a graph parameterization

$\Rightarrow \begin{pmatrix} x \\ y \\ f(x,y) \end{pmatrix}$ $z =$

An image function can be written as a surface



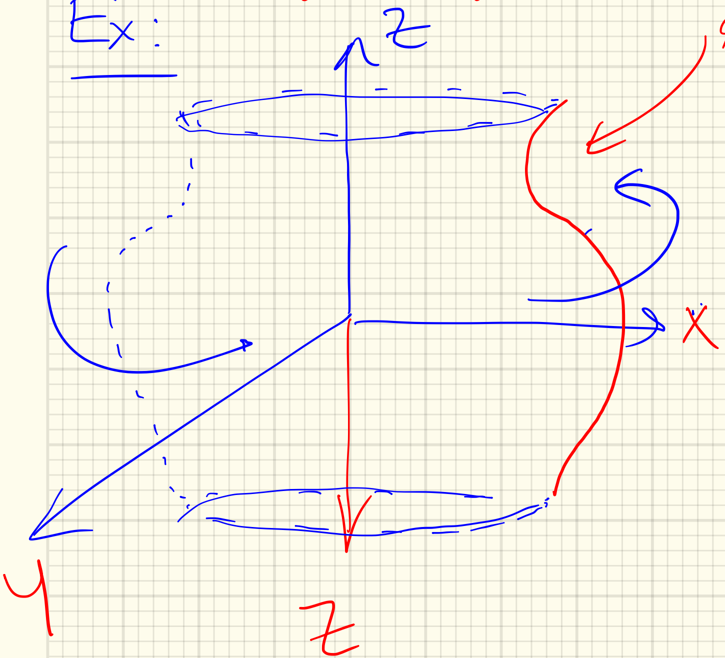
$\begin{pmatrix} x \\ y \\ I(x,y) \end{pmatrix}$

$\Rightarrow$ S_u . ✓ S_v ✓ $\}$ E, F, G .

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix}$$

Surfaces of Revolution:

Ex:



generating curve on the xz plane

$$X = 2 + \cos z$$

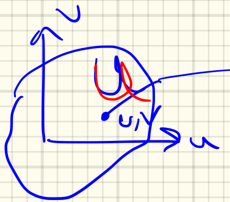
⇝ surface parameterization

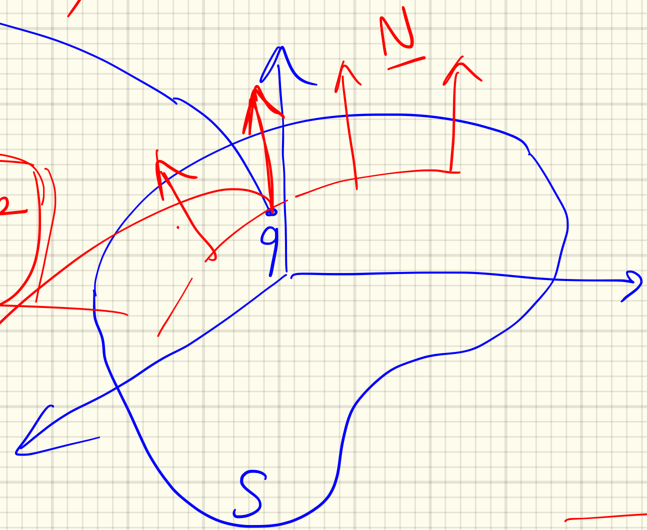
$$S(u,v) = \begin{pmatrix} (2+\cos v)\cos u \\ (2+\cos v)\sin u \\ v \end{pmatrix}$$

Compare this to the parametric form of the cylinder

→ Normal vector at each point $q \in S(\underline{U})$

Normal field $\underline{N}(p) = \dfrac{S_u \times S_v}{\| S_u \times S_v \|}$

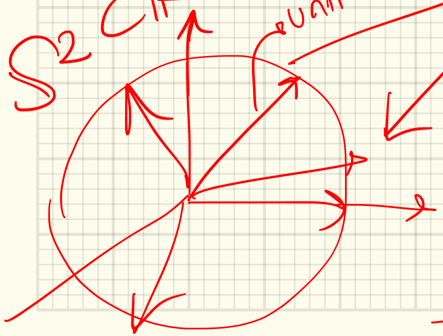




$$\boxed{\underline{N} : S(U) \to S^2}$$

unit vectors

$S^2 \subset \mathbb{R}^3$

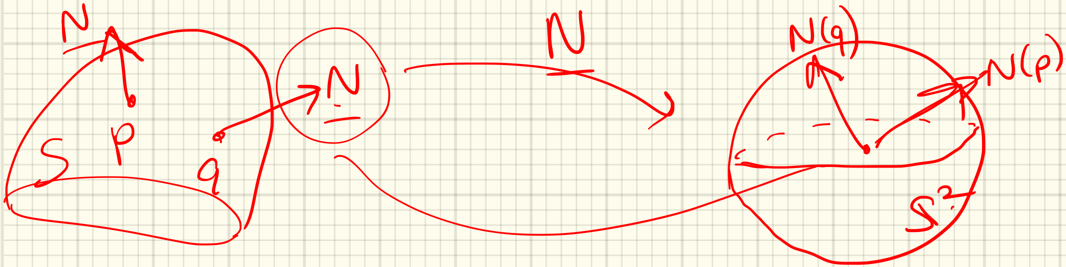


Def: Gauss map is the map $\underline{N} : S \to S^2$

→ Differential of the Gauss map :

$$\boxed{dN_p} : T_p S \longrightarrow T_p(S)$$

N_p : Gauss map : sends a point on the surface to the outward unit normal vector on $\mathbb{S}^2$ (unit sphere)



We want a differentiable normal field.

Möbius strip: → check out the Normal field on the Möbius strip.

Second Fundamental form of a Surface : II_p .

relates to normal vector derivatives $\underline{N}_u$, $\underline{N}_v$.

$$N \cdot N = 1$$

deriv

$$2\underline{N}' \cdot N = 0$$

$$N_u \cdot N = 0$$

$$N_v \cdot N = 0$$

Let $S_{uu} = \dfrac{\partial^2 S}{\partial u^2}$

$S_{uv} = \dfrac{\partial^2 S}{\partial u \partial v}$

$S_{vv} = \dfrac{\partial^2 S}{\partial v^2}$

Def: Derivative operator dN ~ Shape operator

$$S = -dN$$

linear map from tangent plane of the surface S

N_u
N_v $\leftarrow$ $dN : T_p S \longrightarrow T_p S$

is on the tangent plane in a given direction. $\Rightarrow$

→ Define a 2^{nd} Fund. Form of Surface S; w/ coef:

$$e, \quad f, \quad g:$$

Idea: Express N_u, N_v i.t.o. coefficients of the 1^{st} & 2^{nd} fund. form of the surface:

$$\boxed{\underbrace{S_u, S_v}_{T_pS} \longrightarrow \underbrace{N_u, N_v}_{T_pS}}$$

Def: 2^{nd} Fund. Form Coeff:

$$e \triangleq \underline{N} . S_{uu} = -N_u . S_u$$

$$f \triangleq \underline{N} . S_{uv} = -N_u . S_v$$

$$f \triangleq \underline{N} . S_{vu} = -N_v . S_u$$

$$g \triangleq N . S_{vv} = -N_v . S_v$$

$$\underline{N} \cdot \underline{S_u} = 0$$

take deriv. w.r.t. u.

$$N_u \cdot S_u + N . S_{uu} = 0$$

take deriv. w.r.t. v.

$$dN : T_pS \longrightarrow T_pS$$

$$\text{basis } (S_u, S_v) \longrightarrow \text{basis } (N_u, N_v)$$

$$\begin{pmatrix} N_u \\ N_v \end{pmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{\text{Jacobian } dN} \begin{bmatrix} S_u \\ S_v \end{bmatrix}$$

linear map

$$\left.\begin{aligned} N_u &= a_{11} S_u + a_{12} S_v \\ N_v &= a_{21} S_u + a_{22} S_v \end{aligned}\right\}$$

Apply derivative operator dN (or $S = -dN$) to S_u & S_v

$$\underbrace{-S_u \cdot N_u}_{e} = a_{11} \underbrace{(S_u \cdot S_u)}_{E} + a_{12} \underbrace{(S_u \cdot S_v)}_{F}$$

$$e = a_{11} E + a_{12} F$$

$$\underbrace{-N_v \cdot S_v}_{g} = a_{21} \underbrace{(S_u \cdot S_v)}_{F} + a_{22} \underbrace{(S_v \cdot S_v)}_{G}$$

$$g = a_{21} F + a_{22} G$$

$$\rightarrow \underbrace{N_u \cdot S_v}_{-f} = a_{11} \underbrace{S_u \cdot S_v}_{F} + a_{12} \underbrace{S_v \cdot S_v}_{G}$$

$$- f = a_{11} F + a_{12} G$$

$$\underbrace{(N_v \cdot S_u)}_{=} = -f = a_{21} E + a_{22} F$$

$$\left\{ -\begin{bmatrix} e & f \\ f & g \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{\text{Jac } dN} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \right.$$

$$\rightarrow \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{\underline{\underline{A}} = \text{Jac } dN} = - \underbrace{\begin{bmatrix} e & f \\ f & g \end{bmatrix}}_{\underline{\underline{II}}} \cdot \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1}_{\underline{\underline{I}}^{-1}}$$

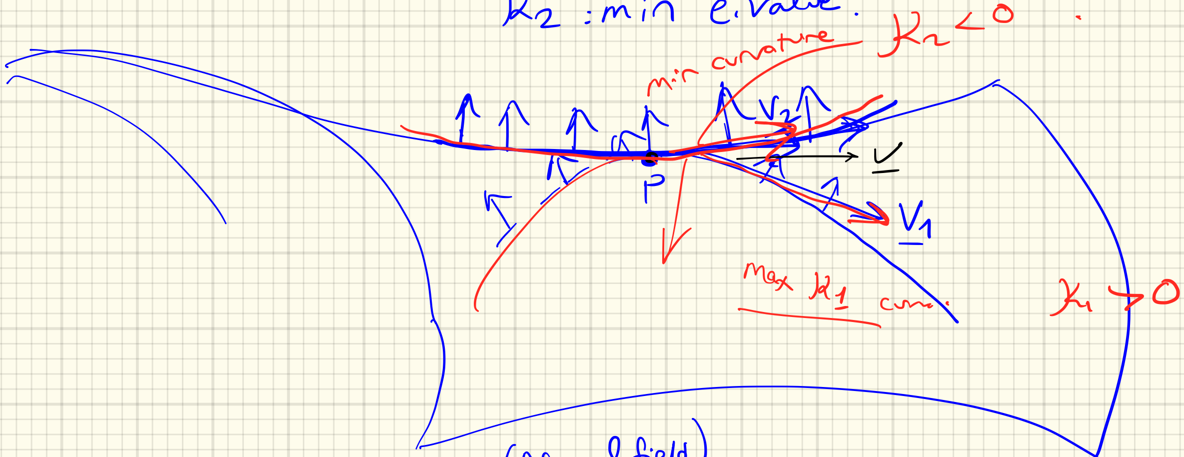
$$\underline{\underline{A}} = - \frac{1}{EG - F^2} \begin{bmatrix} eG - fF & -eF + fE \\ fG - gF & -fF + gE \end{bmatrix}$$

$$\rightarrow \underline{\underline{A}} = \underline{\underline{U}}\, \underline{\underline{\Sigma}}\, V^T$$

$$= \underline{\underline{U}} \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \underline{\underline{V}}^T \longrightarrow \begin{bmatrix} \underline{v_1} & \underline{v_2} \end{bmatrix}$$

k_1 : max e.value

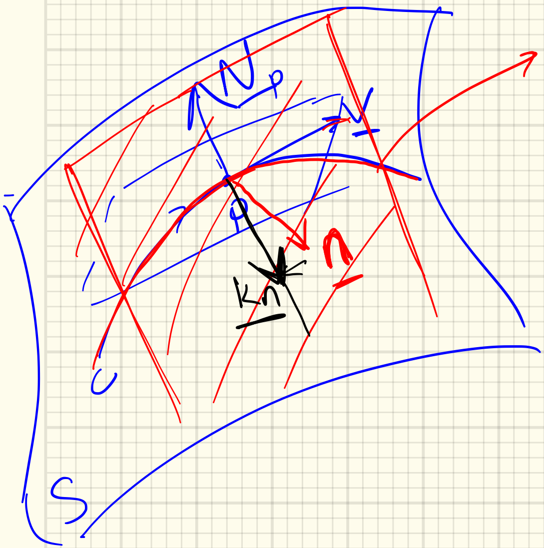
k_2 : min e.value.



★ Differential of the Gauss map (Normal field) $dN : T_pS \rightarrow T_pS$:

measures how $\underline{N}$ pulls away from $N(p)$ in a nbhd $\underline{p}$.

Def: Normal section of the surface S at p in the direction $\underline{v}$.



curve C : intersection of Normal Section (plane) (defined by $\underline{N}$ & $\underline{v}$ (tangent vector)) & surface S

$$\underline{kn} = k \langle \underline{n}, \underline{N} \rangle$$

normal curvature : length of this projection vector (onto N)

Def: The maximum normal curvature K_1 & the minimum " " K_2 called the PRINCIPAL CURVATURES at p on S.

& the corresponding directions (given by the e. vectors of (Jac dN)) are called the principal directions at p.

Def: (Gauss Curvature) of a Surface

$$k_G = k_1 \cdot k_2 \qquad (k_{max} \cdot k_{min})$$

Def: (Mean Curvature)

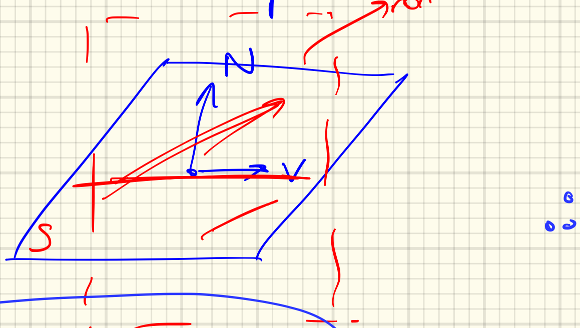
$$k_H = \frac{1}{2}(k_1 + k_2)$$

Def:

$$\boxed{k_{G(p)} = \frac{e \cdot g - f^2}{EG - F^2}} = \frac{\det(\mathrm{II})}{\det(\mathrm{I})} = \det(\mathrm{Jac}\, dN)$$

$$\boxed{k_H = \frac{eG - 2fF + gE}{2(EG - F^2)}} = -\frac{1}{2}\mathrm{Trace}(\overset{\mathrm{Jac}}{dN})$$

Ex: Planar Surface

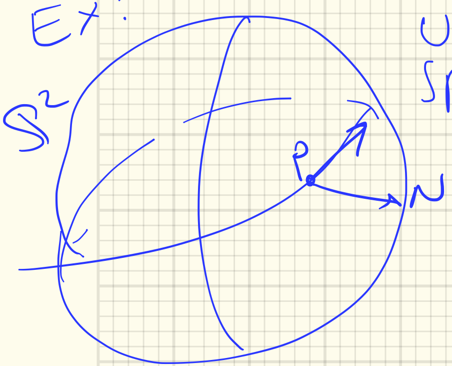


All normal sections (for all directions) are straight lines

$\therefore$ Both principal curvatures $\kappa_{max}\,(\kappa_1)$ & $\kappa_{min}\,(\kappa_2) = 0$.

$\underline{K_G = 0}$, $K_H = 0$.

Ex: Unit Sphere

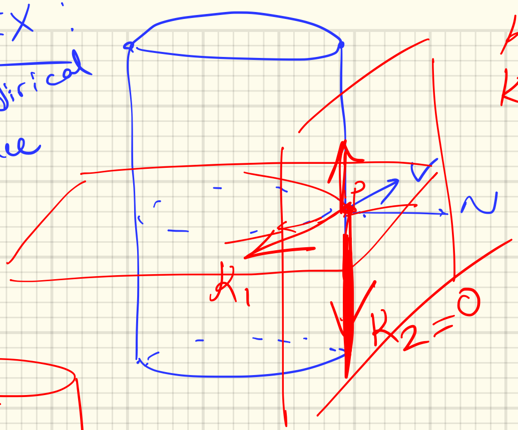


Normal sections thru a point p are circles w/ radius $\underline{1}$

$$\left.\begin{array}{l}\kappa_1 > 0\\ \kappa_2 > 0\end{array}\right\}$$ all normal curvatures $= 1$

$K_G > 0$ $(\kappa_1 \cdot \kappa_2)$ over the sphere $\therefore$ Sphere is an elliptical surface

Ex: Cylindrical surface



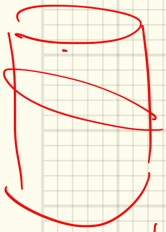
$k_{max} > 0$, k_{min}

$k_1 > 0$, $k_2 = 0$

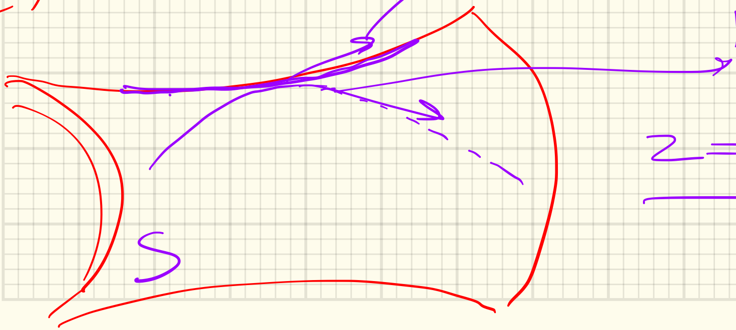
$K_G = 0$

$K_H > 0$

P parabolic point



Ex: Hyperbolic Paraboloid:



$k_2 < 0$

$k_1 > 0$

$K_G < 0$

Hyperbolic point.

$z = y^2 - x^2$

parameterization of S

$$S(u,v) = \begin{pmatrix} u \\ v \\ v^2 - u^2 \end{pmatrix}$$

$$\underline{S_u} = (1, 0, -2u)$$

$$\underline{S_v} = (0, 1, 2v)$$

$$\underline{N} = \frac{S_u \times S_v}{\| \quad \|}$$

$$E = S_u \cdot S_u$$
$$F = S_u \cdot S_v$$
$$G = S_v \cdot S_v$$

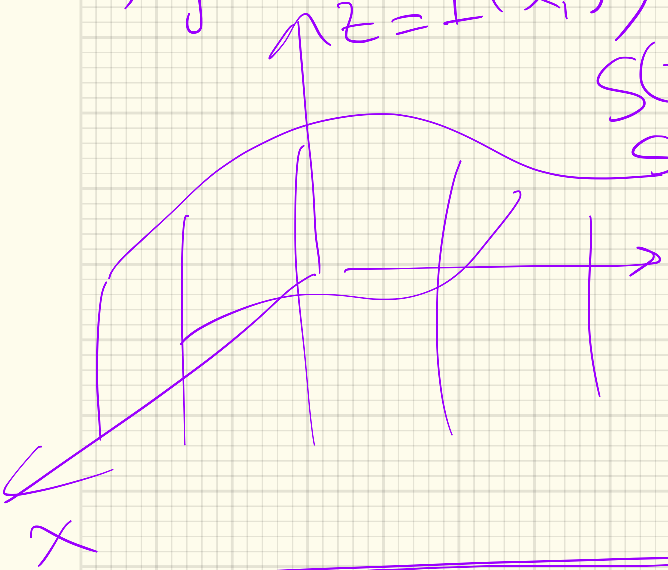
$$e = -N_u \cdot S_u$$
$$f = -N_v \cdot S_u$$
$$g = -N_v \cdot S_v$$

K_G ✓

K_H ✓

View images as a graph surface:

→ eg.



$$S(x,y) = (x, y, I(x,y))$$
$$S(u,v) = (u, v, I(u,v))$$

calculate 1^{st} & 2^{nd} fund. form:

$E, F, G \to \underline{\underline{I}}$

$e, f, g \to \underline{\underline{II}}$

↳ calculate principal curvatures. ✓

★ Fundamental Theorem of Local Theory of Surfaces:

(do Carmo) : First & Second Fundamental forms of a surface uniquely define a regular surface up to a rigid motion!