3D Vision

BLG 634E

23.03.2021

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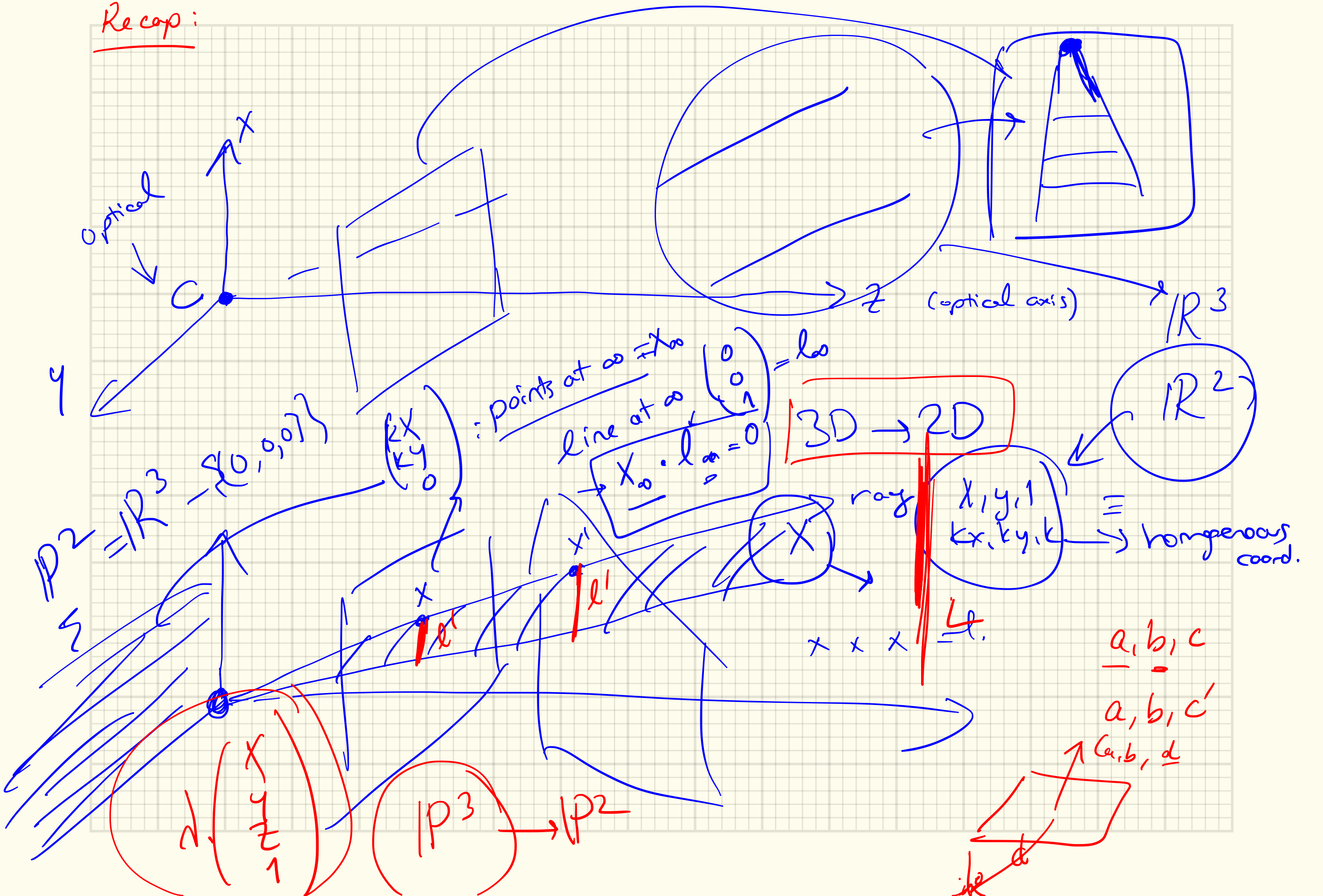
Recap:



$\mathbb{P}^2 = \mathbb{R}^3 - \{(0,0,0)\}$

$\begin{pmatrix} kX \\ kY \\ 0 \end{pmatrix}$: points at $\infty = X_\infty$

line at ∞ $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = l_\infty$

$X_\infty \cdot l_\infty = 0$

3D → 2D

$x, y, 1$

kx, ky, k → homogenous coord.

$\lambda \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$

$\mathbb{P}^3 \to \mathbb{P}^2$

a, b, c

a, b, c'

(a, b, d)

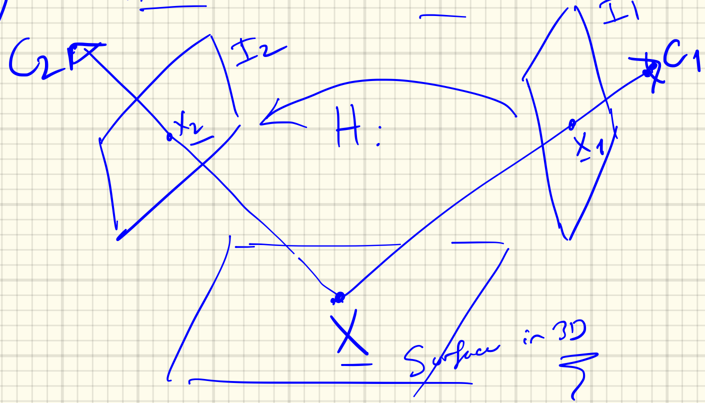
d

1) $\mathbb{P}^3 = \mathbb{R}^4 - \{(0,0,0,0)^T\}$

Projective Geometry: 1) $\mathbb{P}^3 \longrightarrow \mathbb{P}^2$

$$\underbrace{\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}}_{\substack{3D \\ \mathbb{P}^3}} \longrightarrow \underbrace{\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}}_{\substack{2D \\ \mathbb{P}^2}} \to \begin{pmatrix} X/Z \\ Y/Z \end{pmatrix}$$

2) $\underline{\mathbb{P}^2} \longrightarrow \underline{\mathbb{P}^2}$



Projective Transform

$H_{3\times 3}$: $\in PL(3)$

T

Projective Group.

Groups. ?

3) Geometry

Euclidean Transformation $SE(3)$.

3D

Group ?

Some

→ Linear Algebra on Groups.

Linear Algebra ^Group Review (a short one)

: We study linear xformations.
↳ model ← by matrices.

Def: The set of all real $m \times n$ matrices
$n \times n$. matrices $\mathcal{M}(m,n)$ or $\mathcal{M}(n,n)$

Def (Group): A group is a set G w/ an operation "$\circ$" on the elements of G that

1) is closed ; if $g_1, g_2 \in G$ then $g_1 \circ g_2 \in G$.
2) is associative : $g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3$
3) has a unit element e : $e \circ g = g \circ e = g$, $\forall g \in G$
4) is invertible : $\forall g \ \exists g^{-1} \in G$ s.t $g \circ g^{-1} = g^{-1} \circ g = e$

→ Important matrix groups in Computer Vision :

Def: General Linear Group $(GL(n))$ The set of all $n \times n$ (real) non-singular matrices w/ matrix multiplication operation forms a group.

→ Show that all $n \times n$ nonsingular matrices w/ $*$ (matrix) form a group.

1) Let $A, B \in GL(n)$ $\rightarrow C = A * B \in GL(n)$?

2) A, B, C : $A * (B * C) = (A * B) * C$

3) $I_{n \times n}$ ✓

4) $A \rightarrow A^{-1}$

Def: Projective Transformation Group $P(n) = GL(R) // R$

$GL(n)$ matrices known up to a scale factor

$\underline{\underline{H}} \sim k \, \underline{\underline{H}}$

e.g. 3x3 matrix

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & 1 \end{bmatrix}_{3 \times 3} \rightarrow 8 \text{ dof.}$$

Elements of this group are called Homographies or Projective matrices.

Special Linear Group: $A \in GL(n)$ & $\det(A) = \mp 1$.

Affine Group $A(n)$: An affine transformation $L: \mathbb{R}^n \to \mathbb{R}^n$, is defined jointly by a matrix $A \in GL(n)$ & a vector $b \in \mathbb{R}^n$

s.t. $\underline{x} \longrightarrow \underline{\underline{A}}\, \underline{x} + \underline{b}$

$L: \mathbb{R}^n \longrightarrow \mathbb{R}^n$

The set of all such affine transforms is called the affine group $A(n)$: of dimension n.

Q: Is this L map linear? $\underline{\underline{\text{No}}}$ unless $\underline{b} = \underline{0}$.

But we embed this map into a space of 1-dim higher using a homogeneous representation:

$\underline{x} \in \mathbb{R}^n \longrightarrow \begin{bmatrix} \underline{x} \\ 1 \end{bmatrix} \in \mathbb{R}^{n+1}$:

$L: \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$

$\begin{bmatrix} \underline{x} \\ 1 \end{bmatrix} \longrightarrow \underbrace{\begin{bmatrix} \underline{\underline{A}} & \underline{b} \\ \underline{0}^T & 1 \end{bmatrix}}_{\hat{=}\, B \in GL(n+1)} \begin{bmatrix} \underline{x} \\ 1 \end{bmatrix}$

$\underline{0}: \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n$ $\quad A \in GL(n)$, $\underline{b} \in \mathbb{R}^n$ $\longleftarrow$

B fully describes an affine map.

Exercise: Show that it's a group:

1) $$\underbrace{\begin{bmatrix} \underline{\underline{A}}_1 & \underline{b}_1 \\ \underline{0}^T & 1 \end{bmatrix}}_{B_1} \circ \underbrace{\begin{bmatrix} \underline{\underline{A}}_2 & \underline{b}_2 \\ \underline{0}^T & 1 \end{bmatrix}}_{B_2} = \begin{bmatrix} \underline{\underline{A}} & \underline{b} \\ \underline{0}^T & 1 \end{bmatrix}$$

$\longrightarrow$ ✓

3) $$\begin{bmatrix} \underline{\underline{I}}_{n \times n} & \underline{0} \\ \underline{0}^T & 1 \end{bmatrix} = \underline{\underline{I}}_{(n+1),(n+1)}$$ ✓

4) $$\begin{bmatrix} \underline{\underline{A}} & \underline{b} \\ \underline{0}^T & 1 \end{bmatrix}^{-1}$$ exists. $\Longrightarrow$ $$\begin{bmatrix} \underline{\underline{A}}^{-1} & -(\underline{\underline{A}}^{-1}\underline{b}) \\ \underline{0}^T & 1 \end{bmatrix}$$

Def The orthogonal Group $O(n)$

An $n\times n$ matrix A is called orthogonal if it preserves the inner product:

$$\langle \underline{Ax}, \underline{Ay} \rangle = \langle x, y \rangle$$

$$(Ax)^T Ay = x^T \underbrace{A^T A}\, y = x^T \underline{\underline{I}}\, y$$

$$\Rightarrow \boxed{A^T A = I} \Rightarrow A^{-1} = A^T$$

$O(n)$: Set of all orthogonal matrices $\quad O(n) \subset GL(n)$

$$O(n) = \{ R \in GL(n) : R^T R = R R^T = I \}$$

Note: $\det(R) = \mp 1$

Def: Special Orthogonal Group : $SO(n)$: subgroup of $O(n)$ w/ $\det R = +1$

$$SO(n) = \{ R \in GL(n) : R^T R = I \ \& \ \det R = +1 \}$$

For $n=3$, $SO(3)$: 3×3 rotation matrices we'll study.

Def: The Euclidean Group $E(n)$: Affine version of $O(n)$.

$$L: \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$\underline{x} \longrightarrow \underline{\underline{R}}\,\underline{x} + \underline{T}$$

$R \in O(n)$

$T \in \mathbb{R}^n$

$$E(n) \subset A(n)$$

Def: Special Euclidean Group $SE(n)$

$$\left.\begin{array}{l} R \in SO(n) \\ T \in \mathbb{R}^n \end{array}\right\} SE(n)$$

Homogeneous coord.

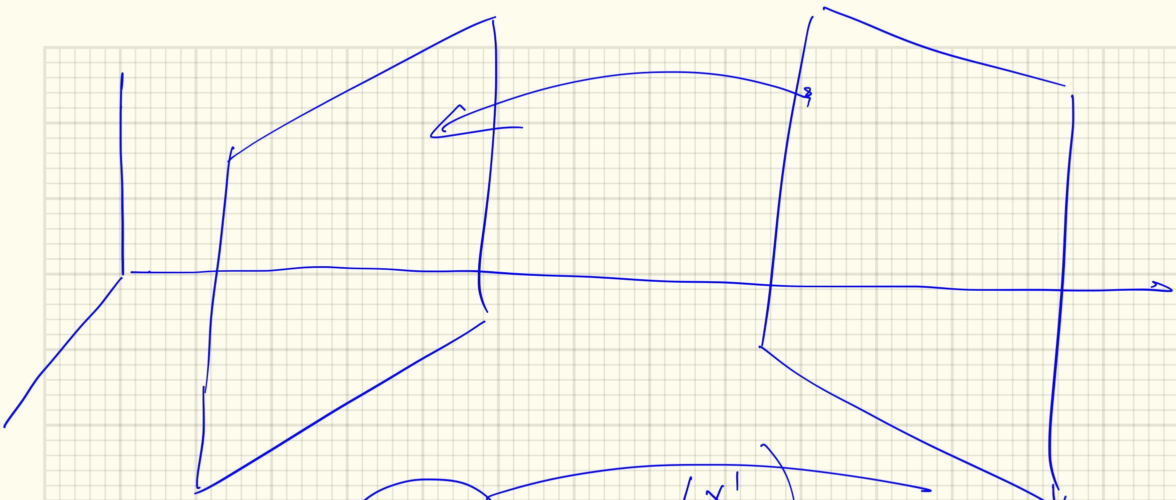
$$\begin{bmatrix} \underline{\underline{R}} & \underline{T} \\ \underline{0}^T & 1 \end{bmatrix} \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$$

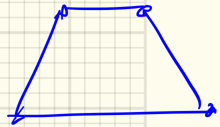
Special case $n=3$

$SE(3)$ represents conventional rigid motion

$R \in SO(3)$: rotation of the rigid body

$T \in \mathbb{R}^3$: translation of the rigid body



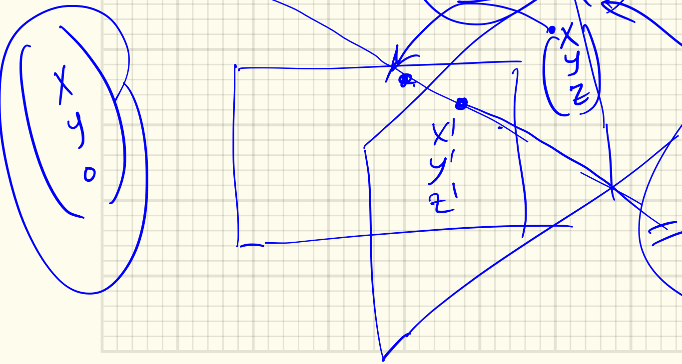


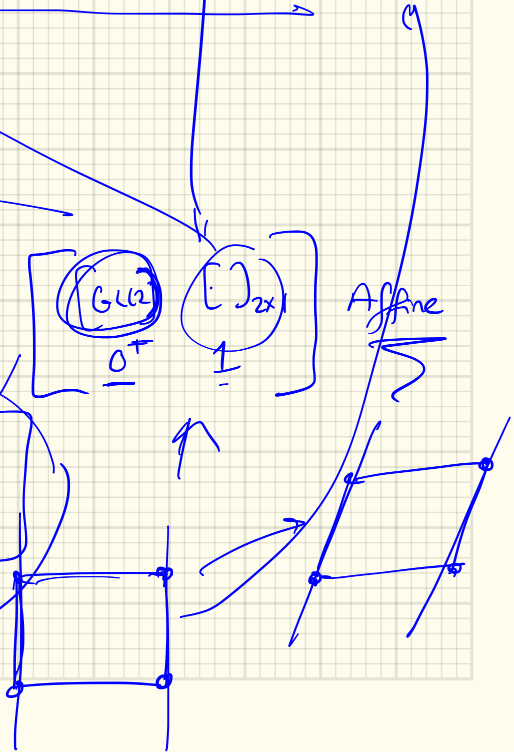
$$H_{3\times 3}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x'\\y'\\z'\end{pmatrix}$$

$$\begin{bmatrix} GL(2) & [\cdot]_{2\times 1} \\ 0^T & 1 \end{bmatrix}$$

Affine

$$\begin{bmatrix} GL(2) & (2\times 1) \\ [\cdot\, v^T \cdot]_{1\times 2} & 1 \end{bmatrix}$$





$A(2)$

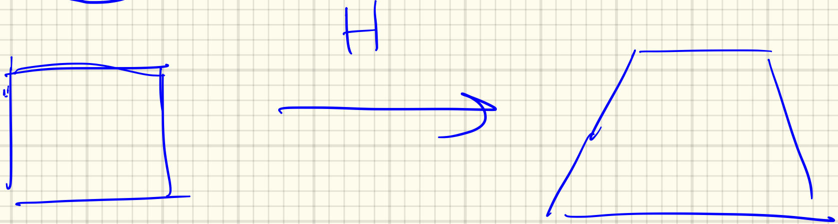
$G(3)$

$$\left[\begin{array}{cc|c} \overbrace{\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}}^{A_{2\times 2}} & & \begin{matrix} b_1 \\ b_2 \end{matrix} \\ \begin{bmatrix} 0 & 0 \end{bmatrix} & & 1 \end{array}\right] \cdot \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x' \\ y' \\ 0 \end{bmatrix} \rightarrow \begin{matrix} x/0 \\ y/0 \end{matrix} \quad ?$$

∞ pt. inf.

$$\underbrace{\begin{bmatrix} A & \begin{matrix} b_1 \\ b_2 \end{matrix} \\ \begin{bmatrix} v_1 & v_2 \end{bmatrix} & 1 \end{bmatrix}}\ \underbrace{\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}} = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \rightarrow \underbrace{\begin{pmatrix} x'/z' \\ y'/z' \end{pmatrix}}$$

finite point.

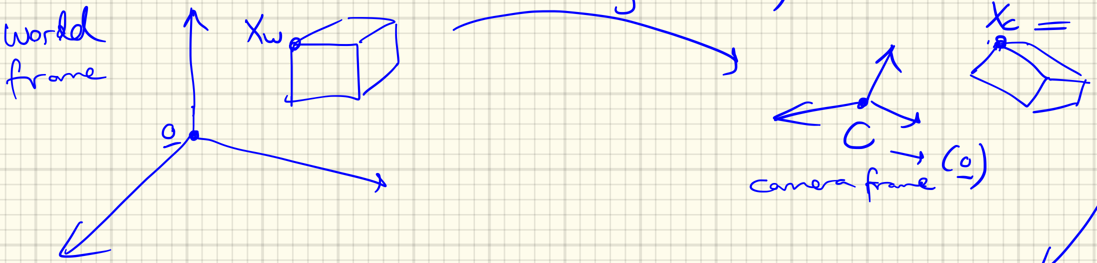


perspective projective.

SO(3) : 3D Rotations

$g = (R, T)$: rigid body motion.

$X_c = R X_w + T$

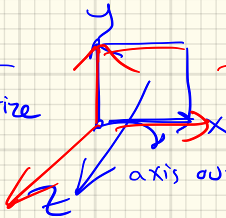


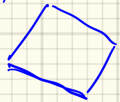
θ : 1 dof : rotation on the plane

$R \in SO(3)$

2D rotations :

easy to parameterize





around a single axis, z axis coming out of the page.

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3D Rotation: not as straightforward 2 rotations. Several possibilities:

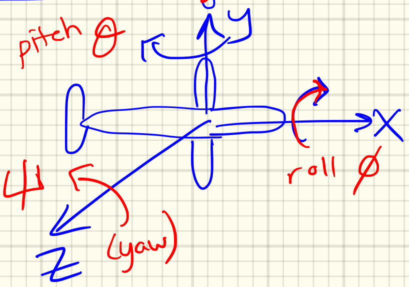
We'll study:

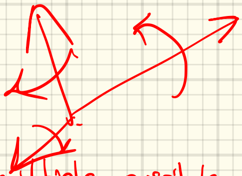
1) Euler Angles : 3 angles.
2) Axis / Angle (Exponential) Representation → SO(3)
3) Quaternions.

Specify rotations in 3D:

$\rightarrow SO(3) = \{ R : 3\times 3 \text{ matrices } R^T R = I \ \& \ \det(R) = +1 \}$

$\boxed{\text{1) Euler Angles:}}$ Stack up 3 coord. axes rotations.





$\boxed{3 \text{ DOF}}$

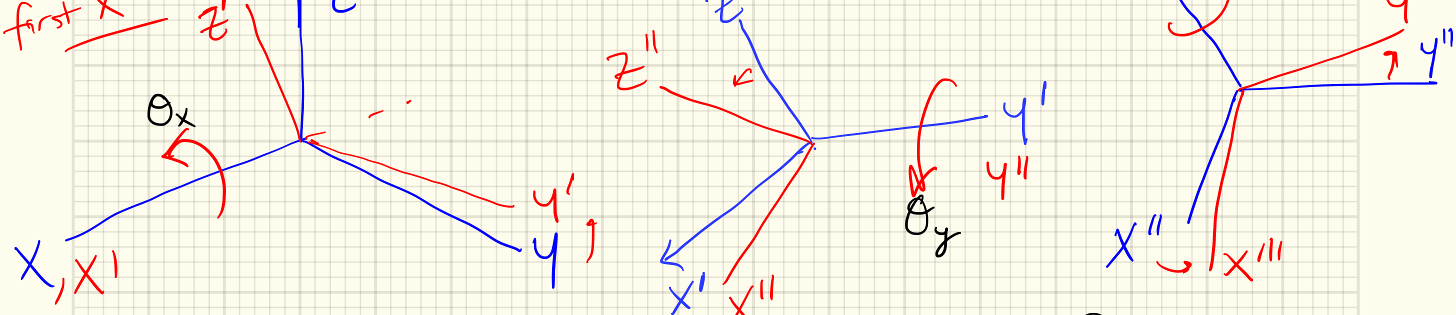
∃ multiple possible sequences of rotation axes.

e.g. $\boxed{ZYX:}$

first X

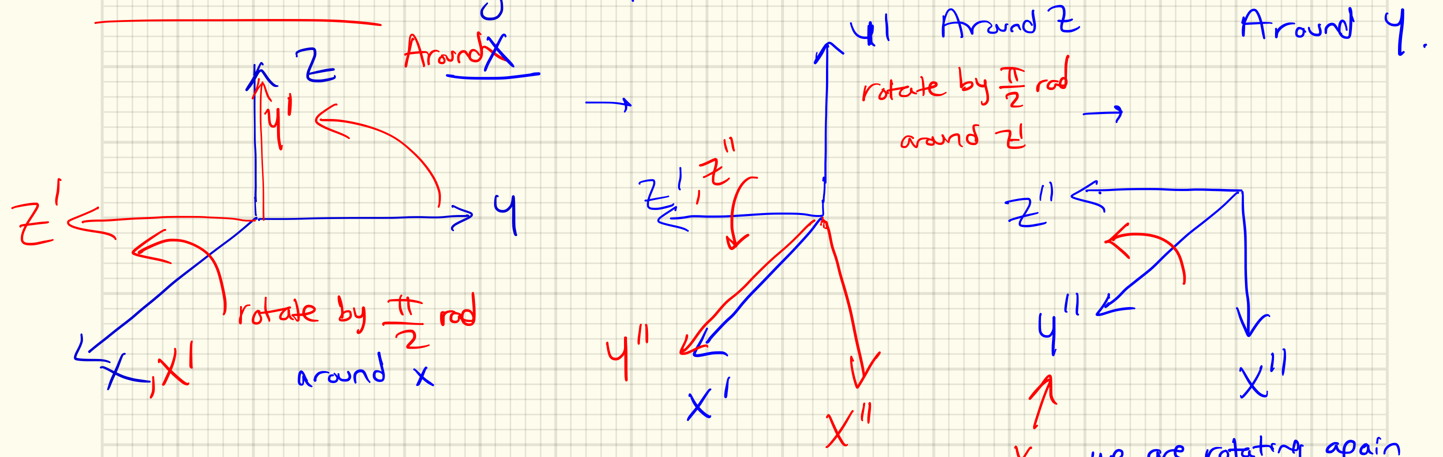
then Y

then Z:



$R(\theta_x, \theta_y, \theta_z) = R_x(\theta_x) \cdot R_{y'}(\theta_y) \cdot R_{z''}(\theta_z) \longrightarrow$

Gimbal Lock: try YZX : (we'll lose 1 dof).



we are rotating again in the same axis

Note: there are many possibilities of Gimbal lock (Aerospace engineers) ← $\boxed{\text{Gimbal Lock example.}}$: we lost 1 dof.

Euler angle Rotation Matrices

$$R_{\theta_x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{bmatrix}, \quad R_{\theta_y} = \begin{bmatrix} \cos\theta_x & 0 & \sin\theta_x \\ 0 & 1 & 0 \\ -\sin\theta_x & 0 & \cos\theta_x \end{bmatrix}, \quad R_{\theta_z} = \begin{bmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or here

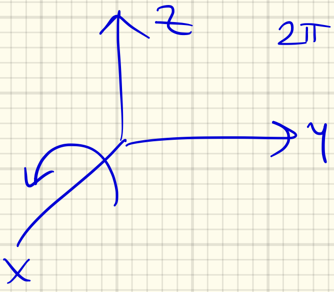
2

· Rotation parameters : does it change continuously ?

Take 2 similar rotations ; eg. $\theta = 359.9°$ rot. around x-axis

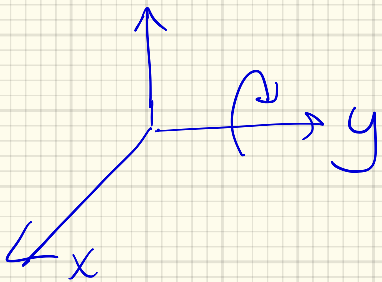
$$\rightarrow R_{\theta x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.9999 & -0.0017 \\ 0 & 0.0017 & 0.9999 \end{bmatrix}$$

2π rotation



Think about rotation by $\theta = 359.9°$ around y axis

$$\rightarrow R_{\theta y} = \begin{bmatrix} 0.999\ldots & 0 & -0.0017 \\ 0 & 1 & 0 \\ 0.0017 & 0 & 0.999\ldots \end{bmatrix}$$



→ Almost the same rotation results → Configurations almost stay the same!

But very different matrix representation !

→ Euler angles ✗. not prefered. representation does not change continuously

→ Gimbal lock problem

$\rightarrow$ Next $so(3)$ Representation Exponential Coord. To do that:

Hat Operator : $\longrightarrow$ Cross product :

Def: Let $\underline{u}, \underline{v} \in \mathbb{R}^3$: $\underline{u} \times \underline{v} \in \mathbb{R}^3 = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$

$\underline{u} = (u_1, u_2, u_3)$

Cross

$\times$ product operator : bilinear in each of its arguments

$$\underline{u} \times (\alpha \underline{v} + \beta \underline{w}) = \alpha\, \underline{u} \times \underline{v} + \beta\, \underline{u} \times \underline{w}$$

$$\underline{u} \times \underline{v} = -\underline{v} \times \underline{u}$$

$$\langle \underline{u} \times \underline{v}, \underline{u} \rangle = 0 = \langle \underline{u} \times \underline{v}, \underline{v} \rangle$$

Fix $\underline{u}$: make a cross product map $\boxed{\hat{\underline{\underline{u}}} : \mathbb{R}^3 \to \mathbb{R}^3}$

$$\mathbb{R}^3, \underline{v} \longrightarrow \hat{\underline{\underline{u}}}\, \underline{v} = \underline{u} \times \underline{v}$$

Def: $\hat{\underline{\underline{u}}} \in \mathbb{R}^{3\times 3}$

$$\hat{\underline{\underline{u}}} \triangleq \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

→ $\hat{\underline{\underline{u}}}$: Skew-symmetric matrix $\hat{\underline{\underline{u}}}^T = -\hat{\underline{\underline{u}}}$

Space of all skew symmetric matrices = little so(3)

$so(3) = \{ \underline{\underline{A}} \in \mathbb{R}^{3\times3} : \underline{\underline{A}}^T = -\underline{\underline{A}} \}$ → exercise check whether so(3) is a group.

Given a $\underline{u}$ vector $\underline{u} \in \mathbb{R}^3$: ⟶ define a $\hat{\underline{\underline{u}}}$ map

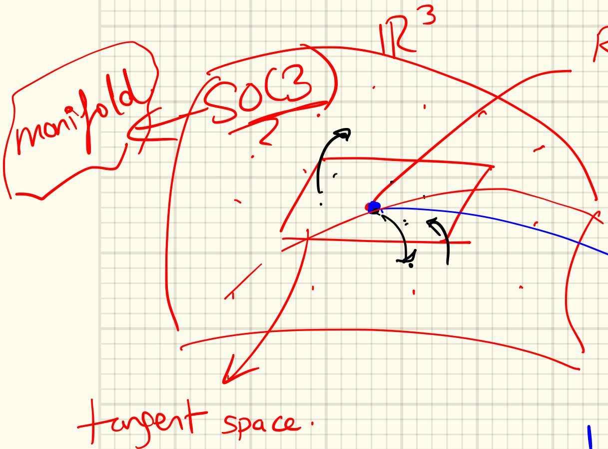
we'll use it

→ in characterizing Rotation matrices.

$$\text{Vector space } \mathbb{R}^3 \longrightarrow \underset{\text{space}}{so(3)} \quad (\text{isomorphic.})$$

$$\underline{u} \longrightarrow \hat{\underline{\underline{u}}}$$

2) Exponential Coordinates to parameterize SO(3) space



$$SO(3) \longleftrightarrow so(3)$$

a family of rotations w/ t a parameter

$$R(t) \in SO(3)$$

$$R(t=0) = I_{3\times3} \quad : \text{identity}$$

$$R(t)\, R^T(t) = I$$

$$\frac{d}{dt}$$

$$\dot{R}(t)\, R^T(t) + R(t)\, \dot{R}^T(t) = 0 \rightarrow (\dot{R} R^T) = -(\dot{R} R^T)^T$$

Let $\hat{\omega} = \dot{R} R^T \in \mathbb{R}^{3\times3}$

$$\hat{\omega} = -\hat{\omega}^T$$

space of skew-symmetric matrices $\leftarrow \hat{\omega} \in so(3) \longrightarrow$

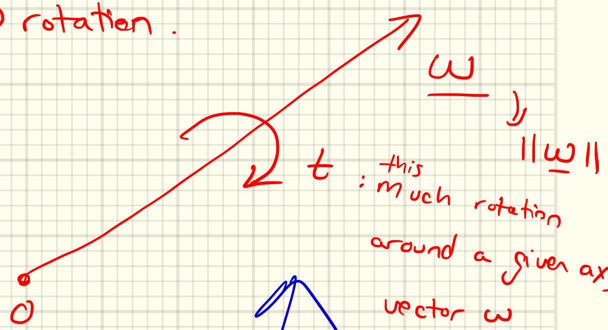
→ ★ Tangent space of the rotation group $\underline{SO(3)}$ is $\underline{so(3)}$

$$\hat{\omega} \cdot R = \dot{R} \underbrace{R^T R}$$

$$\boxed{\hat{\omega} R(t) = \dot{R}(t)} \rightarrow R(t) = e^{\hat{\omega} t} \cdot \underbrace{R(0)}_{= I}$$

$$\boxed{R(t) = e^{\hat{\omega} t}}$$

3D rotation.



t : this much rotation around a given axis vector $\underline{\omega}$.

<u>exercise:</u> Verify that

$e^{\hat{\omega} t}$ is a rotation matrix.

$$R^T R = I \rightarrow \underline{R^{-1} = R^T}$$

$\rightarrow$ Given $\underline{\omega} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \hat{\omega} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$

let $\|\omega\| = 1$

T.S. expansion $e^{\hat{\omega} t} = \underline{\underline{I}} + t\hat{\omega} + \frac{1}{2}(t\hat{\omega})^2 + \dots + \frac{1}{n!}(\hat{\omega} t)^n + \dots$

$$(\hat{\omega})^2 = \begin{bmatrix} x^2-1 & xy & xz \\ xy & y^2-1 & yz \\ xz & yz & z^2-1 \end{bmatrix} = \underbrace{\underline{\omega}\,\underline{\omega}^T - \underline{\underline{I}}}$$

$$(\hat{\omega})^3 = \hat{\omega}(\underline{\omega}\,\underline{\omega}^T - I) = (\underbrace{\underline{\underline{\hat{\omega}}}\,\underline{\omega}}_{\to 0})\underline{\omega}^T - \hat{\omega} = -\hat{\omega}$$

$$(\hat{\omega})^4 = -\hat{\omega}^2$$

... put into T.S. expansion

$$\rightarrow e^{\hat{\omega} t} = \underline{\underline{I}} + \underbrace{\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} + \dots\right)}_{\sin(t)}\hat{\omega} + \underbrace{\left(\frac{1}{2!}t^2 - \frac{t^4}{4!} + \dots\right)}_{1-\cos(t)}\hat{\omega}^2$$

Thm: Rodrigues formula for a Rotation Matrix

(given $\underline{\omega}$ (3D axis of rotation) $\longrightarrow$ compute R), t : amount of rotation (radians angle)

exp map

$$\boxed{R(t) = e^{\hat{\omega} t} = \underline{\underline{I}} + \sin t \, \hat{\omega} + (1 - \cos t)\, \hat{\omega}^2}$$

t : amount of rotation around ω axis

exp map is not (need be) commutative as expected

$$e^{\hat{\omega}_1} \cdot e^{\hat{\omega}_2} \neq e^{\hat{\omega}_2} \cdot e^{\hat{\omega}_1}$$

eg. Cost fn. $\min\limits_{\text{param. } R} E(R) \longrightarrow \frac{\partial}{\partial \omega_i} E(R) = \frac{\partial E}{\partial R} \frac{\partial R}{\partial \omega_i}$

to estimate unknown rotation

if my R was parameterized exp coord ω.

$$\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \xrightarrow{\omega_i} R \checkmark$$

Set up a GD optimizer

$$\boxed{\frac{\partial \omega_i}{\partial t} = -\frac{\partial E}{\partial R} \cdot \frac{\partial R}{\partial \omega_i}}$$

When $\rightarrow t = 2\pi k \quad k \in \mathbb{Z} \rightarrow$ give rise to the same rotation

$$e^{\hat{\omega} t} \longrightarrow e^{\hat{\omega} 2\pi k} = I \longrightarrow$$

(Aside from $0, 2\pi, \ldots$ rotation exp map varies smoothly)

Thm Log of SO(3) Given $R \rightarrow$ find ω.

Reading MaS book Chapter 2

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \longrightarrow \text{by construction} \quad \underbrace{\frac{R - R^T}{2}}_{\text{skew-sym}} = \hat{\omega}$$

$$\theta = t = \operatorname{acos}\left(\frac{\operatorname{trace}(R) - 1}{2} \right)$$

$$\omega = \frac{\theta}{2\sin\theta} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

Note: $\exists$ a single singularity w/ exp-coor. rep.

If $R = I$

$\hat{\omega} = 0 \rightarrow \theta = 0$

$$\omega = \cancel{\frac{\theta}{2\sin\theta} [\;]}$$

for $\theta = 0 \longrightarrow$ not defined!

exclude $\theta = 0$

3) Quaternions : (used a lot by Computer Graphics field)

Complex numbers : $\mathbb{C} = \mathbb{R} + i\,\mathbb{R}$ w/ $\boxed{i^2 = -1}$

$z = x + iy$

Quaternions generalize complex numbers

Set of quaternions :

$\mathbb{H} = \mathbb{C} + j\,\mathbb{C}$ w/ $\boxed{j^2 = -1}$ $\boxed{i \cdot j = -j \cdot i}$

An element of $\mathbb{H}$: $\underline{q} = q_0 + q_1 i + (q_2 + i q_3) j$

$$\begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \underline{q} \in \mathbb{R}^4$$

$$\underline{q} = \underbrace{q_0}_{\text{scalar } s = q_0} + \underbrace{q_1 i + q_2 j + q_3 k}_{\text{vector } \underline{v} = (q_1, q_2, q_3)^T}$$

w/ $k^2 = -1$

$k \triangleq i \cdot j$

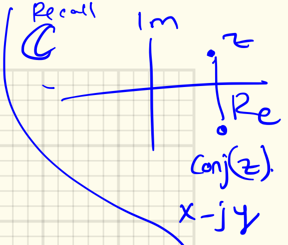
Quaternion Multiplication : $*$

$$\left.\begin{array}{l} \underline{q_1} = (s_1, \underline{v_1}) \\ \underline{q_2} = (s_2, \underline{v_2}) \end{array}\right\} \quad \underline{q_1} * \underline{q_2} \triangleq \Big(\underbrace{s_1 s_2 - \underline{v_1} \cdot \underline{v_2}}_{s},\ \underbrace{s_1 \underline{v_2} + s_2 \underline{v_1} + \underline{v_1} \times \underline{v_2}}_{\underline{v}} \Big)$$

$\underline{q} = (s \quad , \underline{v})$

* Quaternion multiplication is associative

$$q_1 * (q_2 * q_3) = (q_1 * q_2) * q_3$$



Def (Conjugation): $\underline{q} = q_0 + q_1 i + q_2 j + q_3 k$

Conjugate quaternion $\bar{\underline{q}} = q_0 - q_1 i - q_2 j - q_3 k$

Then $\underline{q} * \bar{\underline{q}} = \|\underline{q}\|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2$

$$\underbrace{q^{-1} q} \cdot \bar{q} = \|\underline{q}\|^2 \cdot q^{-1}$$

For a nonzero $\underline{q} \in \mathbb{H}$, $\|\underline{q}\| = 1$, we can define

Def: Inverse quaternion for $\underline{q}$: $\boxed{\underline{q}^{-1} = \dfrac{\bar{\underline{q}}}{\|\underline{q}\|^2}}$