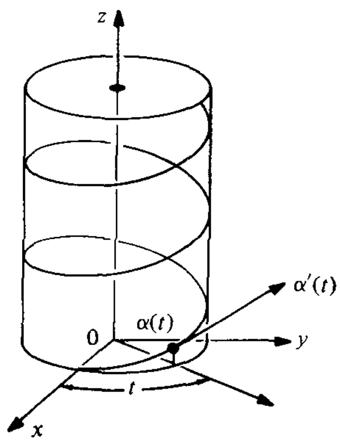
3D Vision
BLG 634E

Professor: Gozde UNAL

Differential Geometry of Curves
An Introduction

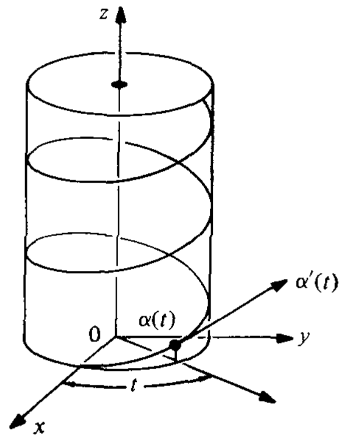
Some figures are due book by Do Carmo: "Differential Geom of Curves and Surfaces"
Some slides are from Scott Schaefer at TAMU, and Jean Gallier's Slides from Upenn
Modified for our course.

Parameterized Curves



Do Carmo, "Differential Geom of Curves and Surfaces"
Figure 1-1.

Parameterized Curves



- Helix:

$$\alpha(t) = (a\cos t, a\sin t, bt)$$

Do Carmo, "Differential Geom of Curves and Surfaces" Figure 1-1.

Intrinsic Properties of Curves

$$x(t) = (\cos(t), \sin(t))$$

$$w(t) = \left(\frac{1-t^2}{1+t^2}, \frac{2t^2}{1+t^2}\right)$$

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$$x(0) = w(0) = (1,0)$$

$$x'(0) = (0,1) \neq (0,2) = w'(0)$$

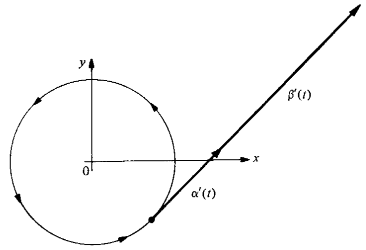
different derivatives!

Parameterized Curves

- Circle

$$\alpha(t) = (\cos t, \sin t)$$

$$\beta(t) = (\cos 2t, \sin 2t)$$



Do Carmo, "Differential Geom of Curves and Surfaces" Figure 1-5.

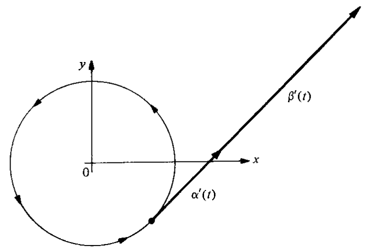
7/50

Parameterized Curves

- Circle

$$\alpha(t) = (\cos t, \sin t)$$

$$\beta(t) = (\cos 2t, \sin 2t)$$



Velocity vector of the second curve is double of the first one

Do Carmo, "Differential Geom of Curves and Surfaces" Figure 1-5.

Arc Length

$$s(t) = \int_a^t \left\| x'(p) \right\| dp$$

- s(t)=t implies arc-length parameterization
- Independent under parameterization!

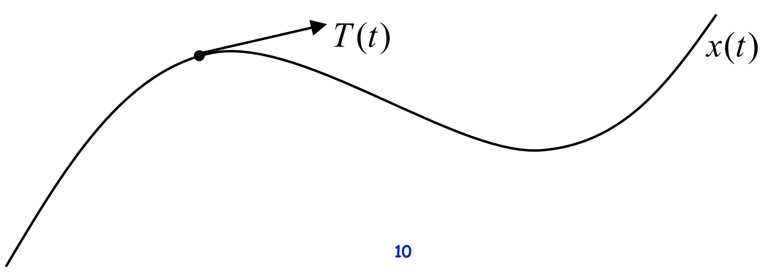
Definition (Tangent vector):For a parameterized differentiable curve:
x: I → R^3,
for each t in I, x'(t) is not equal to zero, there is a well defined straight line, which contains the point x(t) and the vector x'(t).
This line is called **the tangent line to the curve x at t.**

9/50

Frenet Frame (Local Theory of Regular Curves)

- Unit-length tangent

$$T(t) = \frac{x'(t)}{\left\| x'(t) \right\|}$$

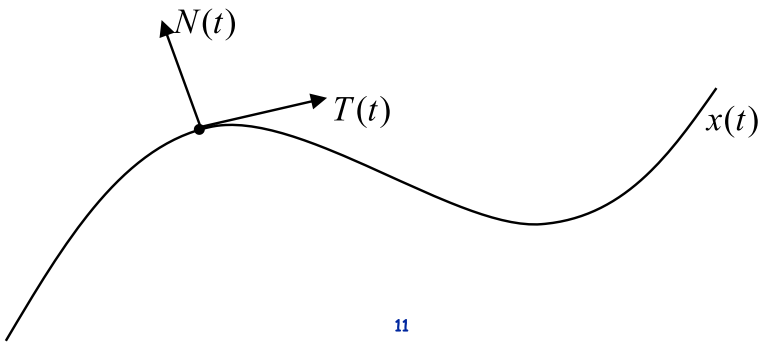


Frenet Frame

- Unit-length tangent
- Unit-length normal

$$T(t) = \frac{x'(t)}{\|x'(t)\|}$$

$$N(t) = \frac{T'(t)}{\|T'(t)\|}$$



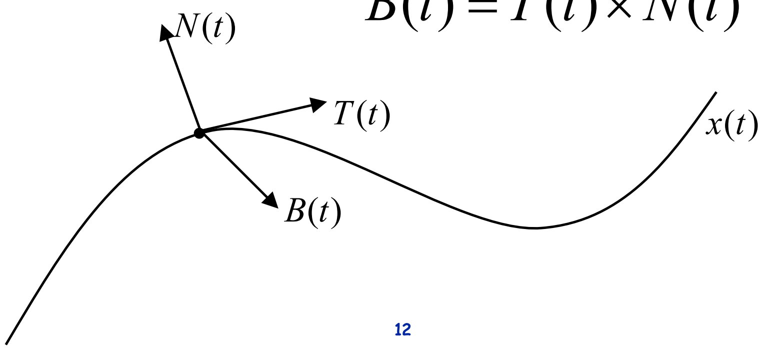
Frenet Frame

- Unit-length tangent
- Unit-length normal
- Binormal

$$T(t) = \frac{x'(t)}{\|x'(t)\|}$$

$$N(t) = \frac{T'(t)}{\|T'(t)\|}$$

$$B(t) = T(t) \times N(t)$$



Frenet Frame

$$T(t)=\frac{x'(t)}{\|x'(t)\|} \qquad N(t)=\frac{T'(t)}{\|T'(t)\|} \qquad B(t)=T(t)\times N(t)$$

- Provides an orthogonal frame anywhere on curve

$$B(t)\cdot T(t)=B(t)\cdot N(t)=T(t)\cdot N(t)=0$$

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$$B(t)\cdot T(t)=B(t)\cdot N(t)=T(t)\cdot N(t)=0$$

Trivial due to cross-product

Frenet Frame

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$$T(t) \cdot T(t) = 1$$

Frenet Frame

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$$T'(t) \cdot T(t) + T(t) \cdot T'(t) = 0$$

Frenet Frame

$$T(t) = \frac{x'(t)}{\|x'(t)\|} \qquad N(t) = \frac{T'(t)}{\|T'(t)\|} \qquad B(t) = T(t) \times N(t)$$

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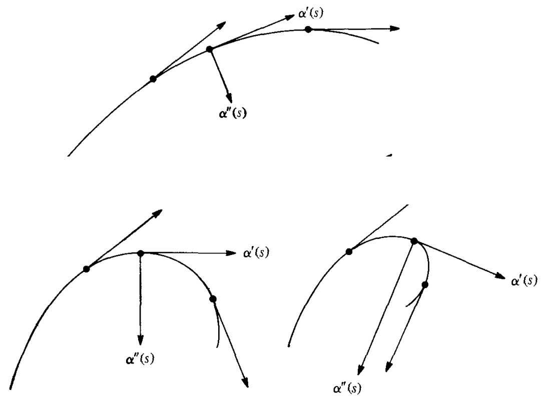
$$T(t) \cdot N(t) = 0$$

Curvature

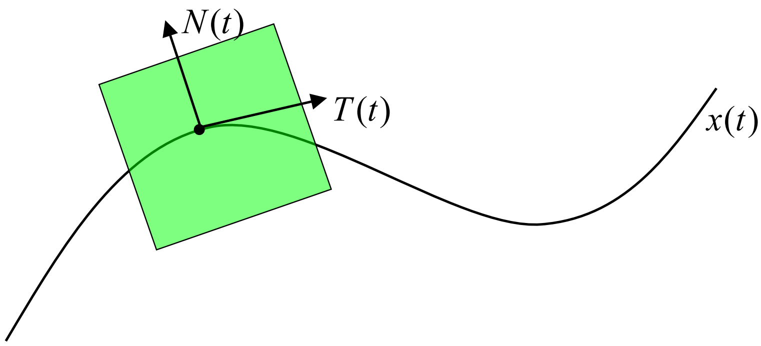
k(s)=|α″|: measure of how rapidly the curve pulls away from the tangent line



Do Carmo, "Differential Geom of Curves and Surfaces" Figure 1-14.

Curvature

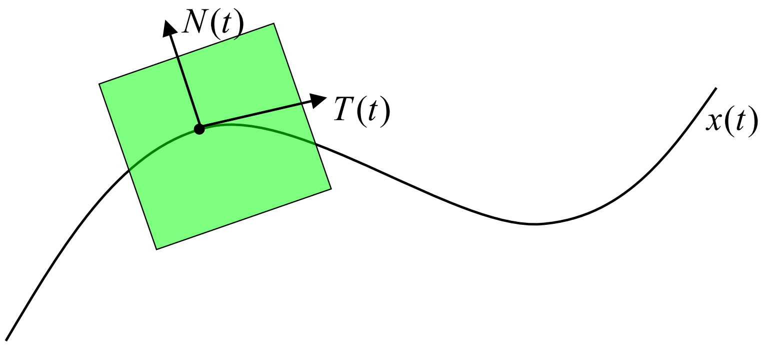
- Measure of how much the curve bends



Curvature

- Measure of how much the curve bends

$$\kappa = \left\| \frac{\partial T}{\partial s} \right\|$$

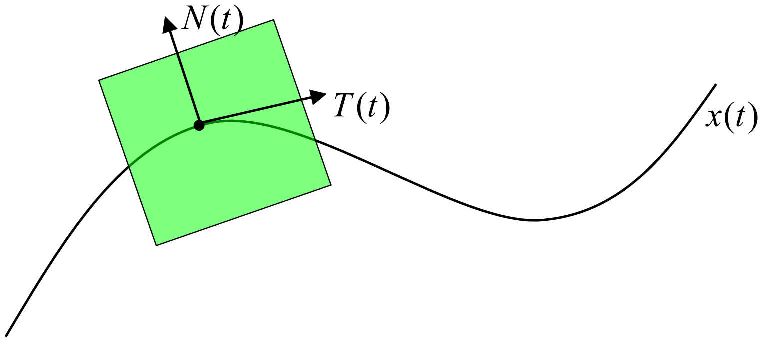


Curvature

- Measure of how much the curve bends

$$\kappa = \left\| \frac{\partial T}{\partial s} \right\|$$

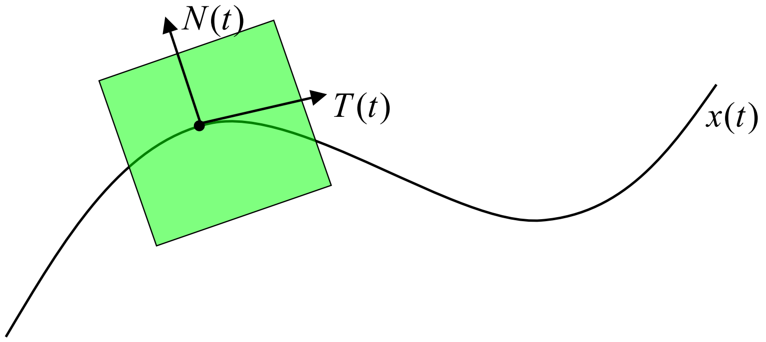
$$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial s} \frac{\partial s}{\partial t}$$



Curvature

- Measure of how much the curve bends

$$\kappa(t) = \frac{\|T'(t)\|}{\|x'(t)\|}$$

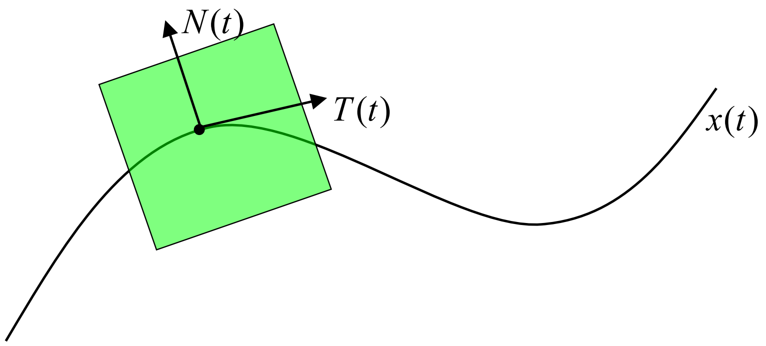


Curvature

- Measure of how much the curve bends

$$\kappa(t) = \frac{\|T'(t)\|}{\|x'(t)\|} = \frac{\|x'(t) \times x''(t)\|}{\|x'(t)\|^3}$$

This last step requires derivation: start from definition of T(t), then continue with deriving to get T'(t) … (If interested, see me).



Curvature

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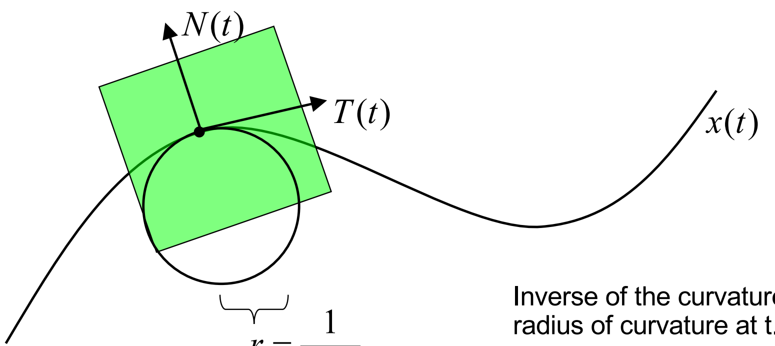
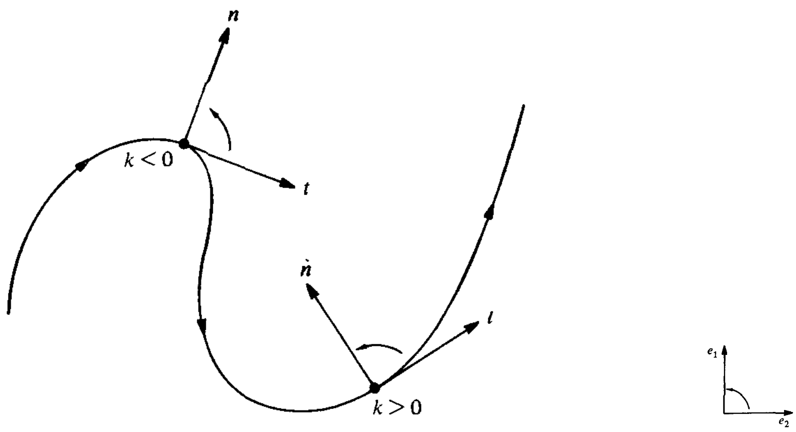


$$r = \frac{1}{\kappa(t)}$$

Inverse of the curvature is called the radius of curvature at t.
E.g. a circle of radius r has radius of curvature = r.

Curvature

- k(s) can be defined as SIGNED

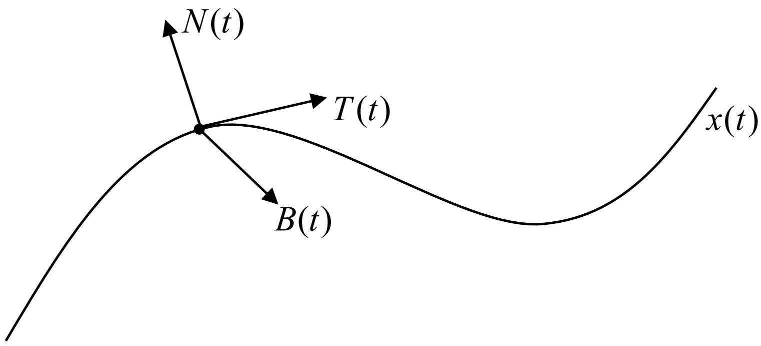


Do Carmo, "Differential Geom of Curves and Surfaces" Figure 1-16.

Torsion

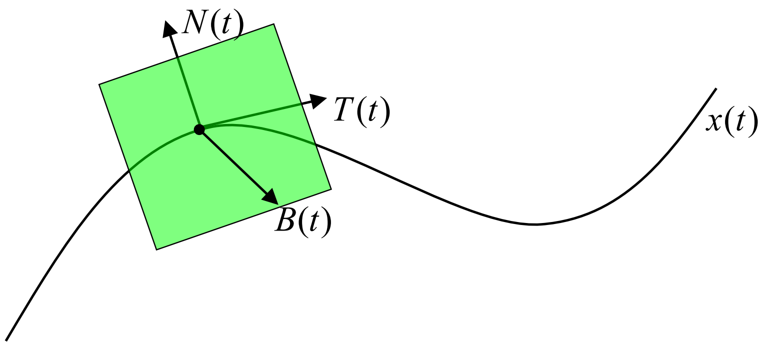
- Measure of how much the curve twists or how quickly the curve leaves the osculating plane

$$\tau(s) = \|B'(s)\|$$



Osculating Plane

- Plane defined by the point x(t) and the vectors T(t) and N(t)
- Locally the curve resides in this plane
- |B'(s)| measures the rate of change of the nbhring osculating planes with the osculating plane at s
- Hence how rapidly the curve pulls away from osc.plane: describes the TORSION of the curve



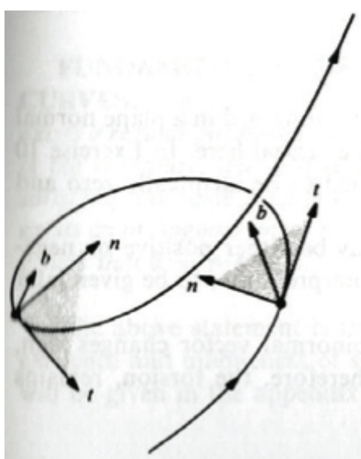
Torsion

- Measure of how much the curve twists or how quickly the curve leaves the osculating plane

$$\tau(s) = \left\| B'(s) \right\|$$

- In terms of the curve C and its derivatives, the torsion of C:

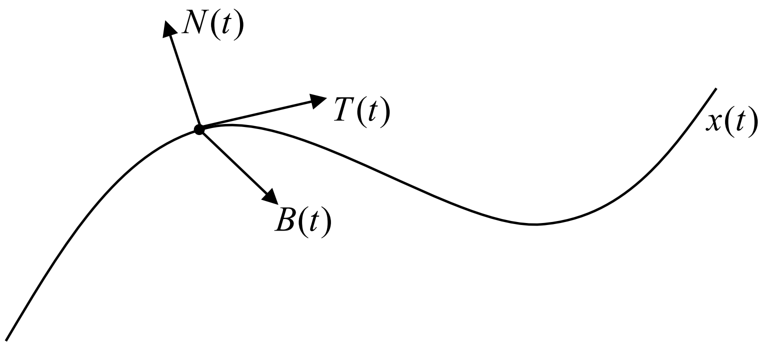
$$\tau(t) = -\frac{(C' \times C'') \cdot C'''}{\left| C' \times C'' \right|^2}$$



Do Carmo, "Differential Geom of Curves and Surfaces" Figure 1-15.

Frenet Equations

- $T'(s) = \kappa(s) N(s)$
- $N'(s) = \tau(s) B(s) - \kappa(s) T(s)$
- $B'(s) = -\tau(s) N(s)$



29/50

29

Frenet Frame

- Physical Intuition:

We can think of a curve in R^3 as being obtained from a straight line by

BENDING (CURVATURE) and

TWISTING (TORSION)

30/50

30

Fundamental Theorem of the Local Theory of Curves

Given differentiable functions k(s)>0 and τ(s), s ∈ I, there exists a regular parameterized curve C: I → R^3 such that s is the arc length, k(s) is the curvature, and τ(s) is the torsion of C.

Moreover, any other curve satisfying the same conditions, differs from C by a rigid motion.

Manfredo P. Do Carmo, "Differential Geometry of Curves and Surfaces", Prentice Hall, 1976.

31/50

Uses of Frenet Frames

- Animation of a camera

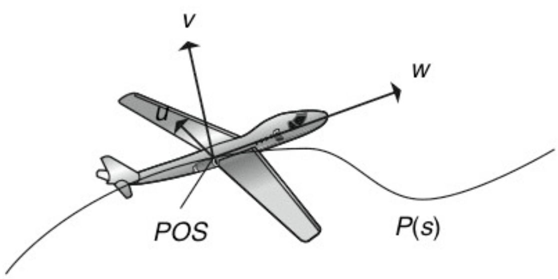


Figure 3.29 Camera-based local coordinate system

Figure from the book: "Computer animation: algorithms and techniques", Rick Parent, 2002

- Skinning a surface along a path

- E.g. Geometric Properties of the 3D Spine Curve

Sotoca J.M., Buendía M., Iñesta J.M., Ferri F.J. (2003) Pattern Recognition and Image Analysis. IbPRIA 2003. Lecture Notes in Computer Science, vol 2652. Springer

E.g. a thoracic scoliosis patient

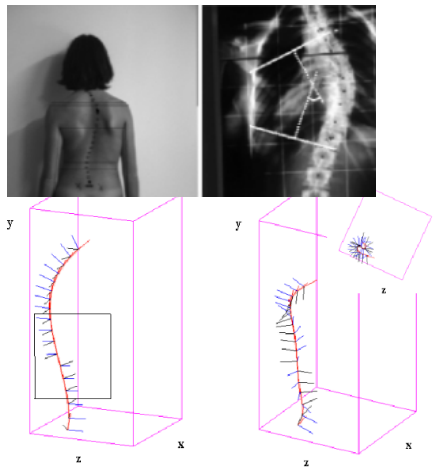


Fig. 3. (Left) A representation of the Frenet frame of a normal spine curve. (Right) Frame evolution for a pathologic spine. In the top-right square, a view from the top is displayed.

Uses of Frenet Frames

- Problems: The Frenet frame becomes unstable or even undefined at inflection points when

$$T'(t) = 0$$

$$T(t) = \frac{x'(t)}{\|x'(t)\|} \qquad N(t) = \frac{T'(t)}{\|T'(t)\|} \qquad B(t) = T(t) \times N(t)$$

Global Properties of Plane Curves

Question: Of all simple closed curves in the plane with a given length L, which one bounds the largest area?

- ISOPERIMETRIC INEQUALITY answers this

Global Properties of Plane Curves

Question: Of all simple closed curves in the plane with a given length L, which one bounds the largest area?

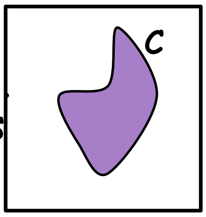
Theorem: (ISOPERIMETRIC INEQUALITY)

Let C be a simple closed plane curve with length L, and let A be the area of the region bounded by C. Then

$$L^2 - 4\pi A >= 0$$

and equality holds if and only if C is a circle.

Global Properties of Plane Curves



JORDAN CURVE THEOREM: Let C be a simple closed curve (i.e. a Jordan curve) in the plane R^2. Then the complement of the image of C consists of two distinct connected components. One of these components is bounded (the interior) and the other is unbounded (the exterior). The image of C is the boundary of each component.

The statement of the Jordan curve theorem seems obvious, but it was a very difficult theorem to prove for general curves!