

BLG634E 3D Vision Lecture on Homography and RANSAC

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IMAGE FORMATION - Perspective Imaging

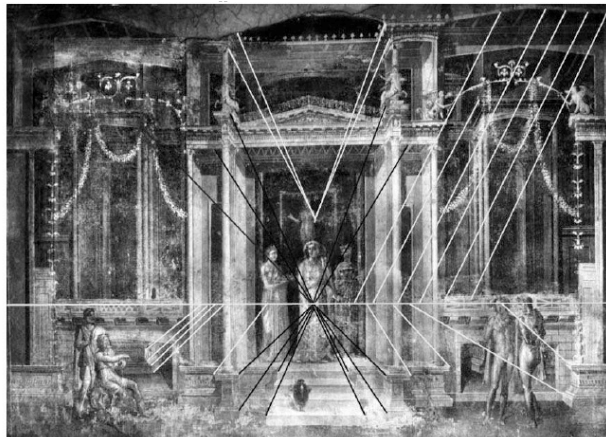


Figure 3.1. Frescoes from the first century B.C. in Pompeii. Partially correct perspective projection is visible in the paintings, although not all parallel lines converge to the vanishing point. The skill was lost during the middle ages, and it did not reappear in paintings until the Renaissance (image courtesy of C. Taylor).

Ma, Soatto et al, Chapter 3

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IMAGE FORMATION - Perspective Imaging

“The Scholar of Athens,” Raphael, 1518

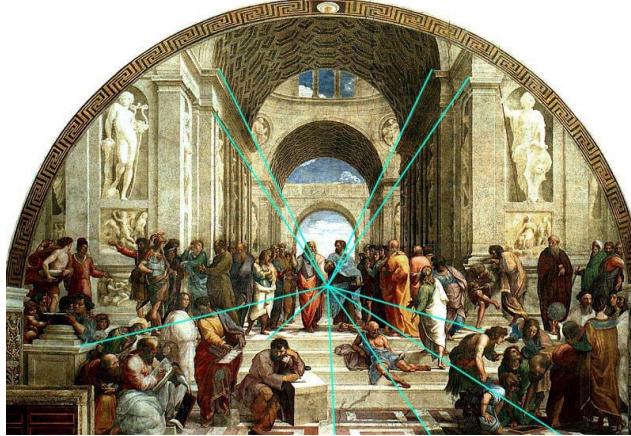


Image courtesy of C. Taylor

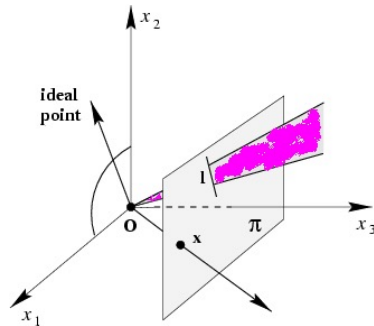
By the end of Renaissance, artists have perfected the techniques on how to paint with correct perspective projection. This is shown vividly and overwhelmingly in paintings from this era.

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Projective Plane

- Points and lines may be obtained by intersecting the set of rays and planes by the plane $x_3 = 1$
- Lines lying in the x_1 - x_2 plane represent ideal points, and the x_1 - x_2 plane represents l_∞ .



- Duality Principle: To any theorem of P^2 projective geometry, there corresponds a dual theorem, with roles of points and lines exchanged
- e.g. A point on a line or line through a point: $l^T x = x^T l = 0$
- e.g. Intersection of two lines, or points: $x = l \times l'$ or $l = x \times x'$

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Projective Transformation

- ❑ In a view of geometry (F. Klein): geometry is the study of properties invariant under groups of transformations
- ❑ From this view: 2D projective geometry: study of properties of the projective plane P^2 under a group of transformations called Projectivity.
- ❑ Def: **Projectivity**: An invertible mapping from points in P^2 to points in P^2 that maps lines to lines
- ❑ Projectivities form a group

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Central Projection

- ❑ Fig 1.3 Projection of rays through a common point (center of projection) defines a mapping from one plane to another
- ❑ Central projection maps points on one plane to points on another plane.
- ❑ It also maps lines to lines (consider a plane through the projection center which intersects the two planes π and π')
- ❑ Therefore, **central projector is a projectivity** and represented by a linear mapping of homogeneous coordinates

$$\bullet \rightarrow x' = Hx$$

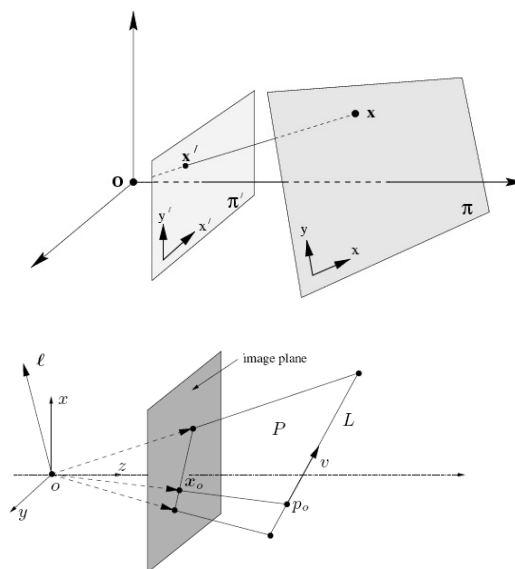


Figure 3.10. Perspective image of a line L in 3-D. The collection of images of points on the line forms a plane P . Intersection of this plane and the image plane gives a straight line ℓ which is the image of the line.

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The Projective Geometry of 1D (Extra Material)

□ **The Cross Ratio:** Basic projective invariant of P^1

□ **Def:** Given 4 points x_i (homogeneous coord):

$$\text{Cross}(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4) = \frac{|\bar{x}_1 \bar{x}_2| |\bar{x}_3 \bar{x}_4|}{|\bar{x}_1 \bar{x}_3| |\bar{x}_2 \bar{x}_4|} \quad |\bar{x}_i \bar{x}_j| = \det \begin{bmatrix} x_{i1} & x_{j1} \\ x_{i2} & x_{j2} \end{bmatrix}$$

if $\bar{x}' = H_{2 \times 2} \bar{x}$ then $\text{Cross}(\bar{x}'_1, \bar{x}'_2, \bar{x}'_3, \bar{x}'_4) = \text{Cross}(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)$

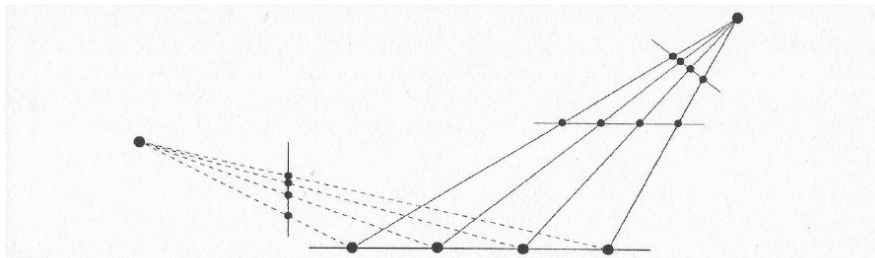


Fig. 1.8. Projective transformations between lines. There are four sets of four collinear points in this figure. Each set is related to the others by a line-to-line projectivity. Since the cross ratio is an invariant under a projectivity, the cross ratio has the same value for all the sets shown.

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Projective Transformation

□ **Examples of Projective transformation**

□ $x' = Hx$

□ **Projective transform between two images**

- induced by a world plane
- with the same camera center (camera rotating about its center)

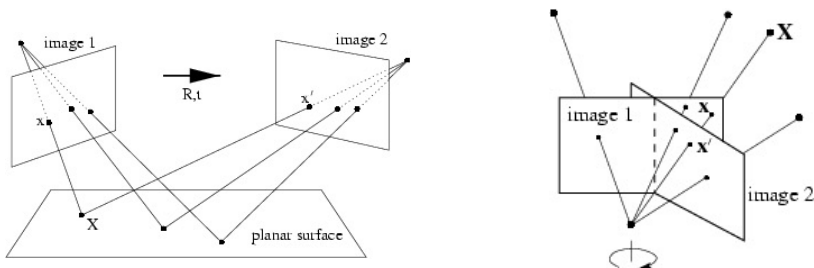


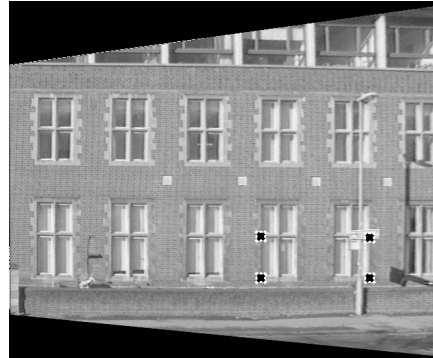
Fig 1.5. Hartley, Zisserman

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Removing Perspective Distortion

- ❑ Shape is distorted under perspective imaging (windows not rectangular on the left)



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Fig 1.4. Hartley,Zisserman

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Projective Transformation

- ❑ Distortions arising under central projection
- ❑ a. *Similarity*: circle, square imaged as circle, square. Angles are preserved
- ❑ b. *Affine*: circle \rightarrow ellipse, parallel lines still parallel
- ❑ c. *Projective*: Parallel lines \rightarrow converging lines. Tiles closer to the camera have a larger image than those farther away.

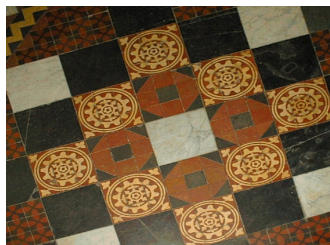
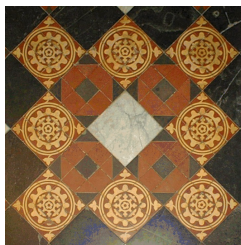


Fig 1.6. Hartley,Zisserman

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Projective Transformation

□ Decomposition of a projective transformation

$$H = H_S H_A H_P = \begin{bmatrix} sR & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} K & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{v}^T & v \end{bmatrix} = \begin{bmatrix} A & \mathbf{t} \\ \mathbf{v}^T & v \end{bmatrix} \quad (1.16)$$

with A a non-singular matrix given by $A = sRK + \mathbf{t}\mathbf{v}^T$, and K an upper-triangular matrix normalized as $\det K = 1$. This decomposition is valid provided $v \neq 0$, and is unique if s is chosen positive.

- Each of the above matrices has the "essence" of a transformation of that type (e.g. Similarity, affine and projective)

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2D Transformation Groups and Invariants





Group	Matrix	Distortion	Invariant properties
Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, order of contact : intersection (1 pt contact); tangency (2 pt contact); inflections (3 pt contact with line); tangent discontinuities and cusps. cross ratio (ratio of ratio of lengths).
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallelism, ratio of areas, ratio of lengths on collinear or parallel lines (e.g. midpoints), linear combinations of vectors (e.g. centroids). The line at infinity, l_∞ .
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle. The circular points, I, J (see section 1.7.3).
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area

Table 1.1. Geometric properties invariant to commonly occurring planar transformations. The matrix $A = [a_{ij}]$ is an invertible 2×2 matrix, $R = [r_{ij}]$ is a 2D rotation matrix, and (t_x, t_y) a 2D translation. The distortion column shows typical effects of the transformations on a square. Transformations higher in the table can produce all the actions of the ones below. These range from Euclidean, where only translations and rotations occur, to projective where the square can be transformed to any arbitrary quadrilateral (provided no three points are collinear).

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from [Hartley,Zisserman] book

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The Projective Geometry of 3D

Transformation	Matrix	# DoF	Preserves	Icon	Group	Matrix	Distortion
translation	$\begin{bmatrix} \mathbf{I} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}_{3 \times 4}$	3	orientation		Projective 15 dof	$\begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^T & v \end{bmatrix}$	
rigid (Euclidean)	$\begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}_{3 \times 4}$	6	lengths		Affine 12 dof	$\begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$	
similarity	$\begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}_{3 \times 4}$	7	angles		Similarity 7 dof	$\begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$	
affine	$\begin{bmatrix} \mathbf{A} \\ \mathbf{0}^T & 1 \end{bmatrix}_{3 \times 4}$	12	parallelism		Euclidean 6 dof	$\begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$	
projective	$\begin{bmatrix} \tilde{\mathbf{H}} \\ \mathbf{0}^T & 1 \end{bmatrix}_{4 \times 4}$	15	straight lines				

Table 2.2 Hierarchy of 3D coordinate transformations. Each transformation also preserve the properties listed in the rows below it, i.e., similarity preserves not only angles but also parallelism and straight lines. The 3×4 matrices are extended with a fourth $[\mathbf{0}^T \ 1]$ row to

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Homography

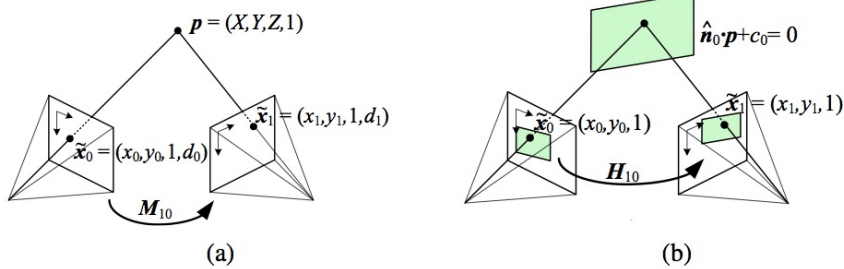


Figure 2.12 A point is projected into two images: (a) relationship between the 3D point coordinate $(X, Y, Z, 1)$ and the 2D projected point $(x, y, 1, d)$; (b) planar homography induced by points all lying on a common plane $\hat{\mathbf{n}}_0 \cdot \mathbf{p} + c_0 = 0$.

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Homography Estimation



Figure 5.13. Homography between the left and middle images is determined by the building facade on the top, and the ground plane on the bottom. The right image is the warped image overlaid on the first image based on estimated homography H . Note that all points on the reference plane are perfectly aligned, whereas points outside the reference plane are offset by an amount that is proportional to their distance from the reference plane.

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Panoramic Mosaicing



Fig. 8.9. **Planar panoramic mosaicing.** Eight images (out of thirty) acquired by rotating a camcorder about its centre. The thirty images are registered (automatically) using planar homographies and composed into the single panoramic mosaic shown. Note the characteristic "bow tie" shape resulting from registering to an image at the middle of the sequence.

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from [Hartley,Zisserman] book

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Homography Estimation: Direct Linear Transform

Simple linear algorithm to estimate H

$$Hx_i = x'_i$$

Then

$$x'_i \times Hx_i = 0.$$

$$Hx_i = \begin{pmatrix} h^1 \top x_i \\ h^2 \top x_i \\ h^3 \top x_i \end{pmatrix}$$

with

$$x'_i = (x'_i, y'_i, w'_i)^\top$$

$$x'_i \times Hx_i = \begin{pmatrix} y'_i h^3 \top x_i - w'_i h^2 \top x_i \\ w'_i h^1 \top x_i - x'_i h^3 \top x_i \\ x'_i h^2 \top x_i - y'_i h^1 \top x_i \end{pmatrix}$$

$$\rightarrow \begin{bmatrix} 0^\top & -w'_i x_i^\top & y'_i x_i^\top \\ w'_i x_i^\top & 0^\top & -x'_i x_i^\top \\ -y'_i x_i^\top & x'_i x_i^\top & 0^\top \end{bmatrix} \begin{pmatrix} h^1 \\ h^2 \\ h^3 \end{pmatrix} = 0. \quad (3.1)$$

$$h = \begin{pmatrix} h^1 \\ h^2 \\ h^3 \end{pmatrix}, \quad H = \begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix} \quad (3.2)$$

These equations have the form $A_i h = 0$, where A_i is a 3×9 matrix.

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(Chapter 3, [HZbook])

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Homography Estimation: The Basic DLT

3.1 The Direct Linear Transformation (DLT) algorithm

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Objective

Given $n \geq 4$ 2D to 2D point correspondences $\{x_i \leftrightarrow x'_i\}$, determine the 2D homography matrix H such that $x'_i = Hx_i$.

Algorithm

- (i) For each correspondence $x_i \leftrightarrow x'_i$ compute the matrix A_i from (3.1). Only the first two rows need be used in general.
- (ii) Assemble the n 2×9 matrices A_i into a single $2n \times 9$ matrix A .
- (iii) Obtain the SVD of A (section A3.3(p556)). The unit singular vector corresponding to the smallest singular value is the solution h . Specifically, if $A = UDV^\top$ with D diagonal with positive diagonal entries, arranged in descending order down the diagonal, then h is the last column of V .
- (iv) The matrix H is determined from h as in (3.2).

Algorithm 3.1. The basic DLT for H (but see algorithm 3.2(p92) which includes normalization).

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Homography Estimation: The Basic DLT

- If more than 4 point correspondences are given
 - Over-determined system
- If the positions of points are exact, rank of A still 8 then we still have 1-dim null space for h
- If measurements of image coord are not exact ("noise") there will not be an exact solution to the overdetermined system $Ah=0$ other than $h=0$
- Then search for an approx soln (instead of $Ah=0$)
 - Minimize a cost function with a constraint on h

$$\min \|Ah\| \quad \text{s.t. } \|h\|=1$$

- The constraint avoids $h=0$
- The solution is the (unit) eigenvector of $A^T A$ with least eigen value

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DLT: Direct Linear Transform revisited

- *DLT not invariant* to image coordinate transformation: e.g. a similarity transformation

- Let T and T' be two similarity transformations:

$$\begin{aligned}\tilde{x}_i &= T x_i \\ \tilde{x}'_i &= T' x'_i\end{aligned}$$

- Substitute in $x' = Hx$ $\tilde{x}' = T'HT^{-1}\tilde{x}$

- This changes the minimization of the algebraic error

$$\|Ah\| = s\|\tilde{A}\tilde{h}\|$$

with the norm constraint $\|h\|=1$ now becomes $\|\tilde{h}\|=1$ (Recall h is the vector entries of homography $H_{3 \times 3}$ elements)

- Solution: Normalize the data before applying the DLT to handle the arbitrary origin and scale of image coordinates \rightarrow Algo 3.2

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Normalized DLT for 2D homographies

Objective

Given $n \geq 4$ 2D to 2D point correspondences $\{x_i \leftrightarrow x'_i\}$, determine the 2D homography matrix H such that $x'_i = Hx_i$.

Algorithm

- (i) **Normalization of x :** Compute a similarity transformation T , consisting of a translation and scaling, that takes points x_i to a new set of points \tilde{x}_i such that the centroid of the points \tilde{x}_i is the coordinate origin $(0, 0)^T$, and their average distance from the origin is $\sqrt{2}$.
- (ii) **Normalization of x' :** Compute a similar transformation T' for the points in the second image, transforming points x'_i to \tilde{x}'_i .
- (iii) **DLT:** Apply algorithm 3.1(p73) to the correspondences $\tilde{x}_i \leftrightarrow \tilde{x}'_i$ to obtain a homography \tilde{H} .
- (iv) **Denormalization:** Set $H = T'^{-1}\tilde{H}T$.

Algorithm 3.2. *The normalized DLT for 2D homographies.*

Note: Data normalization is an essential step of DLT!

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Robust Estimation of the Homography

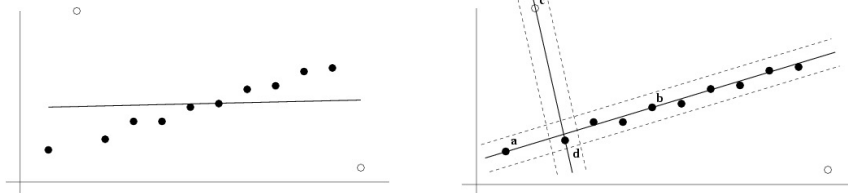
- ❑ Until now, we inherently assumed that point correspondences $x_i \leftrightarrow x'_i$ had only source of error in the measurements of point position (with a Gaussian distribution)
- ❑ **Mismatched points**
 - \rightarrow outliers to Gaussian error distribution
 - Will severely affect the estimated homography
 - Should be identified
- ❑ **Goal:** determine a **set of inliers** from the presented correspondences so that the homography can be estimated in an optimal manner
- ❑ \rightarrow This is **robust estimation!** Robust (tolerant) to outliers (measurements following a possibly unmodelled error distribution)

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Robust Estimation

- Robust Line Estimation: Regression problem Fig 3.7. [HZbook]

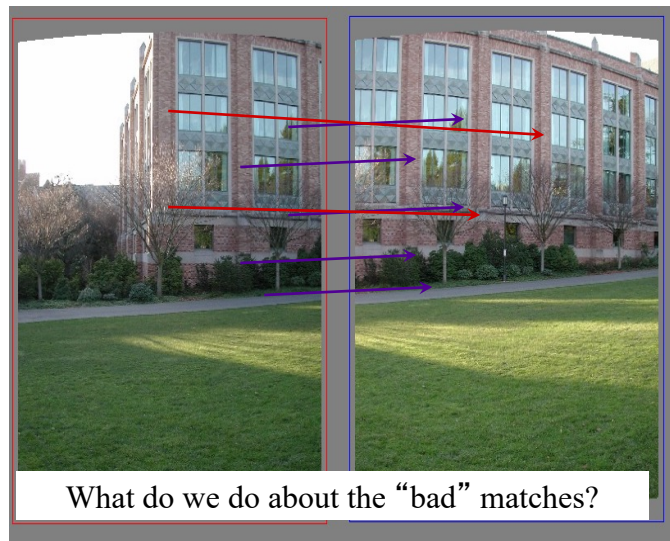


- Inliers: Solid points, Outliers: Open points
- RANSAC Idea: Select two points randomly \rightarrow they define a line.
- Measure the support by no. of points that lie within a distance threshold
- Repeat this random selection a number of times
- The line with the most support is the robust fit.
- Inliers: points within the threshold distance (consensus set)
- If a point is an outlier, that line will not gain much support!

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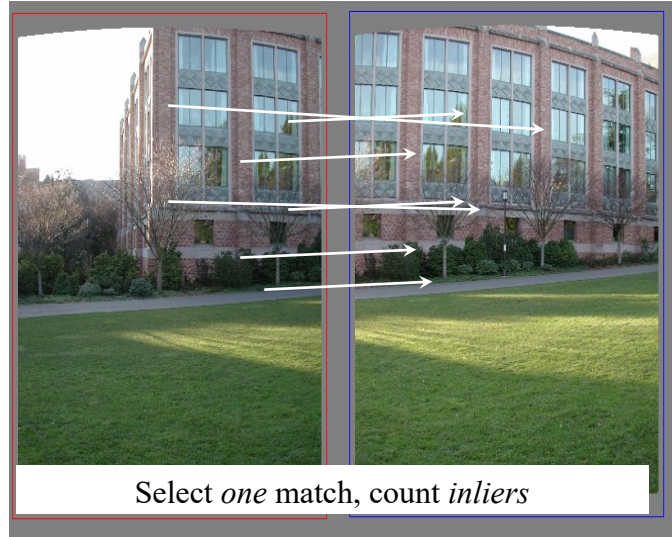
Matching features



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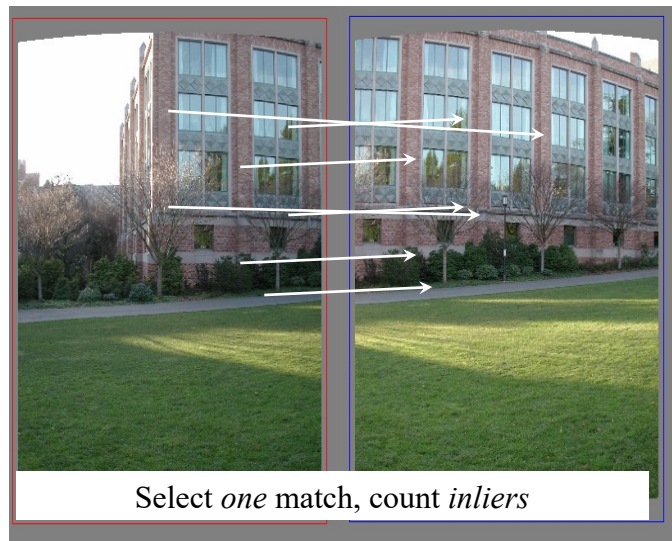
Random Sample Consensus



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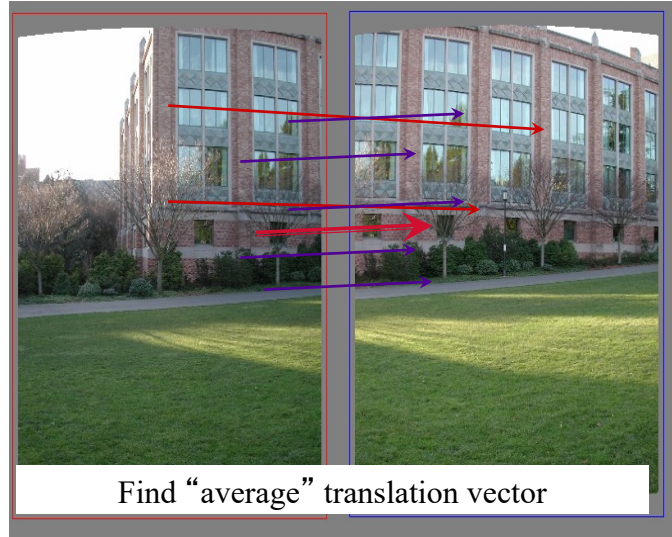
Random Sample Consensus



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Least squares fit



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RANSAC for estimating homography

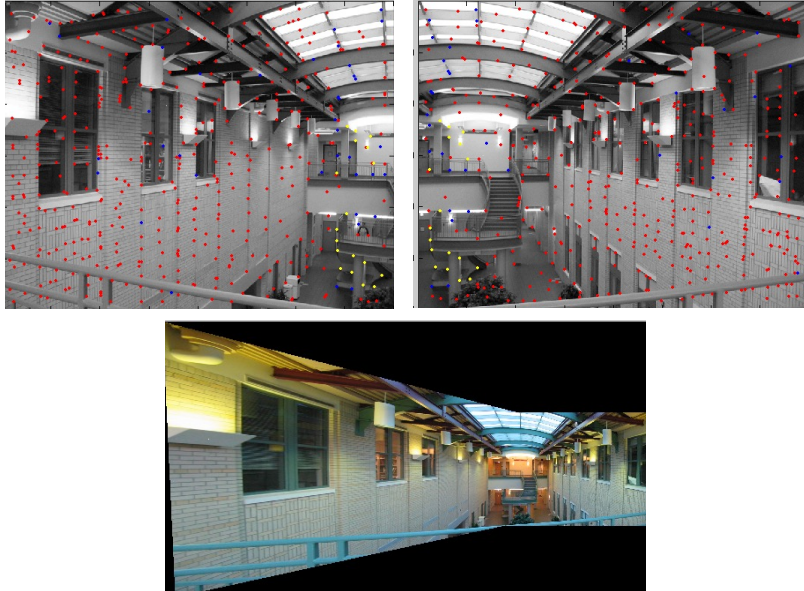
□ RANSAC loop:

1. Select four feature pairs (at random)
2. Compute homography H (exact)
3. Compute *inliers* where $SSD(p_i', H p_i) < \epsilon$
4. Keep largest set of inliers (in case of ties, choose the solution with lowest std dev of inliers)
5. Re-estimate H on all of the inliers

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RANSAC



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RANSAC: Random Sample Consensus

- ❑ Generally: we want to fit a model to the data (e.g. a line in the previous ex, or a homography to point correspondences)
- ❑ Random sample consists of a **minimal subset of the data** sufficient to determine the model (e.g. two points for a line)
- ❑ If the model is a planar homography, and the data a set of 2D point correspondences, then the minimal subset contains 4 correspondences.
- ❑ As Fischler and Bolles put it [Fischler-81]
"The RANSAC procedure is opposite to that of conventional smoothing techniques: Rather than using as much of the data as possible to obtain an initial solution and then attempting to eliminate the invalid points, RANSAC uses as small an initial dataset as feasible and enlarges this set with consistent data when possible"
- ❑ → Use RANSAC in homography estimation

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Robust Estimation: General RANSAC Algo

Objective

Robust fit of a model to a data set S which contains outliers.

Algorithm

- (i) Randomly select a sample of s data points from S and instantiate the model from this subset.
- (ii) Determine the set of data points S_i which are within a distance threshold t of the model. The set S_i is the consensus set of the sample and defines the inliers of S .
- (iii) If the size of S_i (the number of inliers) is greater than some threshold T , re-estimate the model using all the points in S_i and terminate.
- (iv) If the size of S_i is less than T , select a new subset and repeat the above.
- (v) After N trials the largest consensus set S_i is selected, and the model is re-estimated using all the points in the subset S_i .

Algorithm 3.4. The RANSAC robust estimation algorithm, adapted from [Fischler-81]. A minimum of s data points are required to instantiate the free parameters of the model. The three algorithm thresholds t , T , and N are discussed in the text.

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RANSAC Parameters

1. Distance threshold t ? in practice, chosen empirically

Table 3.2 [HZbook] probabilistic t values: $t^2 = c\sigma$, where Normal(0, σ) measurement error

2. How many sample sets? N samples: number of samples sufficiently high to ensure with a probability p (usually $p=0.99$) that at least 1 sample set is free from outliers

- w : prob that a point is an inlier $\rightarrow \epsilon = 1-w$ prob of outlier
- At least N selections (each has s points) are required: $(1-w^s)^N = 1-p$

$$N = \log(1-p) / \log(1 - (1-\epsilon)^s). \quad (3.18)$$

- $N = \infty$, sample.count = 0.
- While $N > \text{sample.count}$ Repeat
 - Choose a sample and count the number of inliers.
 - Set $\epsilon = 1 - (\text{number of inliers}) / (\text{total number of points})$
 - Set N from ϵ and (3.18) with $p = 0.99$.
 - Increment the sample.count by 1.
- Terminate.

Algorithm 3.5. Adaptive algorithm for determining the number of RANSAC samples.

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(Chapter 4 [HZbook])

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RANSAC Parameters

3. How large is an acceptable consensus set?

Rule of thumb: terminate if

size of consensus set \sim estimated number of inliers
believed to be in the data

Ex:

ϵ : prob of outliers = 0.2 (20%)

For a total of n data points:

$T = (1 - \epsilon)^n = 0.8^n$: estimated no of inliers in the dataset

T would be a good estimate for an acceptable number of
data points expected in the consensus set

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Automatic Computation of a Homography

Objective

Compute the 2D homography between two images.

Algorithm

- (i) **Interest points:** Compute interest points in each image.
- (ii) **Putative correspondences:** Compute a set of interest point matches based on proximity and similarity of their intensity neighbourhood.
- (iii) **RANSAC robust estimation:** Repeat for N samples, where N is determined adaptively as in algorithm 3.5:
 - (a) Select a random sample of 4 correspondences and compute the homography H .
 - (b) Calculate the distance d_{\perp} for each putative correspondence.
 - (c) Compute the number of inliers consistent with H by the number of correspondences for which $d_{\perp} < t = \sqrt{5.99} \sigma$ pixels.Choose the H with the largest number of inliers. In the case of ties choose the solution that has the lowest standard deviation of inliers.
- (iv) **Optimal estimation:** re-estimate H from all correspondences classified as inliers, by minimizing the ML cost function (3.8-p78) using the Levenberg-Marquardt algorithm of section A4.2(p569).
- (v) **Guided matching:** Further interest point correspondences are now determined using the estimated H to define a search region about the transferred point position.

The last two steps can be iterated until the number of correspondences is stable.

Algorithm 3.6. *Automatic estimation of a homography between two images using RANSAC.*
(Chapter 3, [HZbook])

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Computation of Homography

via RANSAC

- ❑ Motion (rotation about the camera center) between views

- ❑ Corners detected in both

- ❑ Left Figure: All putative point correspondences

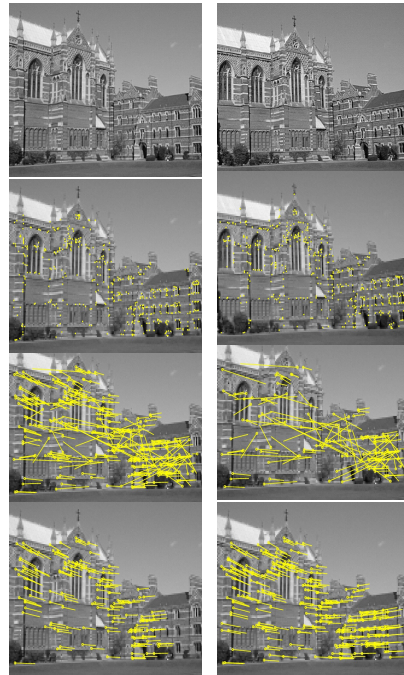
Right Figure: Outliers

- ❑ Left: Inliers

- ❑ Right: Final set of inliers after iterations in RANSAC

Fig 3.9 [HZbook]

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Error functions in estimating H

- ❑ DLT algo minimizes the norm $\|Ah\|$. The vector $\epsilon = \|Ah\|$ is called the residual vector.

- ❑ Each correspondence $x_i \leftrightarrow x'_i$ contributes to a partial error vector ϵ_i toward the full error: called the algebraic error vector

- ❑ The norm of this distance is the algebraic distance:

$$d_{\text{alg}}(x'_i, Hx_i)^2 = \|\epsilon_i\|^2 = \left\| \begin{bmatrix} \mathbf{0}^\top & -w'_i x_i^\top & y'_i x_i^\top \\ w'_i x_i^\top & \mathbf{0}^\top & -x'_i x_i^\top \end{bmatrix} \mathbf{h} \right\|^2. \quad (3.4)$$

- ❑ Error for the complete set:

$$\sum_i d_{\text{alg}}(x'_i, Hx_i)^2 = \sum_i \|\epsilon_i\|^2 = \|Ah\|^2 = \|\epsilon\|^2. \quad (3.5)$$

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Error functions in estimating H (geometric error)

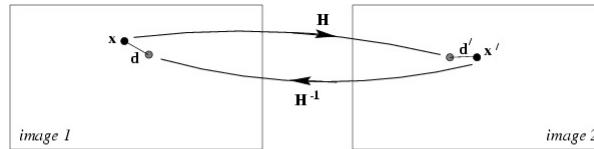


Fig 3.2: Symmetric transfer error (geometric error) minimizes

$$\sum_i d(x_i, H^{-1}x'_i)^2 + d(x'_i, Hx_i)^2. \quad (3.7)$$

- ❑ image measurement errors occur in both images
x and x' are measured image coordinates (noisy points)
- ❑ 1st (2nd) term: transfer error in the 1st(2nd) image
- ❑ d : Euclidean image distance

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Iterative Minimization Methods

Minimize the geometric errors like the symmetric error through iterative minimizations such as

Newton's method

Levenberg-Marquardt method

Setting up the iterative minimization:

- ❑ Recall symmetric cost function:

$$\min \sum_i d(x_i, H^{-1}x'_i)^2 + d(x'_i, Hx_i)^2$$

- ❑ Define a function $f : h \mapsto (H^{-1}x'_1, \dots, H^{-1}x'_n, Hx_1, \dots, Hx_n)$

- ❑ An initial estimate for h can be found from e.g. Algo 3.2

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Assignments

- ❑ Reading: Hartley-Zisserman book Chap 3 (Homography estimation)

- ❑ Next time: Zhang's paper, A flexible New technique for Camera Calibration
Study the paper/technical report by Zhang on his Camera Calibration technique, on the board

- ❑ Camera Calibration Implementation: Check Bouget's Website, his camera calibration links, etc
<http://www.vision.caltech.edu/bouquetj/>
 - Gather components of the code
 - Build the grid
 - Experiment and Test it!

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