19.09.2022

YZV 231E

Probability Theory & Stats

Week 1

Gü.

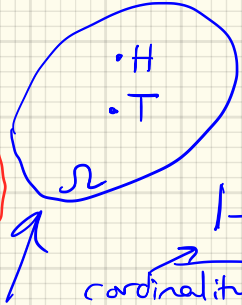
We learn what it takes to set up a probabilistic model

① Sample Space : contains all the outcomes of a random experiment

ex : Random experiment : Flipping a coin :

Outcomes → Heads (H)
→ Tails (T)

Sample Space Ω



$|\Omega| = 2$

cardinality of Ω

② Probability Laws: → should satisfy certain properties → AXIOMS of PROBABILITY

describe my beliefs about which outcomes are more likely to occur compared to other outcomes.

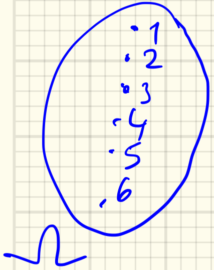
③ Define Events : sets of outcomes : subsets of the sample space Ω.

④ Calculate probability of Events :

Sample Space: We execute a particular experiment K

We list all the outcomes

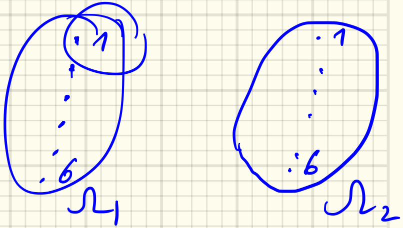
ex: Experiment: rolling a dice



A = { getting "4" on the dice }

$$P(A) = \frac{1}{6}$$

$$|\Omega| = 6$$



ex: Experiment : Rolling two die :

1st dice, 2nd dice

$$\Omega: \begin{array}{l}(1,1), (1,2), ----- (1,6)\\ (2,1), (2,2), --- (2,6)\\ \vdots \qquad\qquad\qquad\qquad \vdots\\ (6,1)\ (6,2)\ --\ (6,6)\end{array}$$

$$|\Omega| = 36$$

$$\Omega = \Omega_1 \times \Omega_2$$

cartesian product

$$\Omega = \{(\overset{\Omega_1}{1}, \overset{\Omega_2}{1}), (1,2) ....., (1,6), (2,1), ...\ (6,6)\}$$

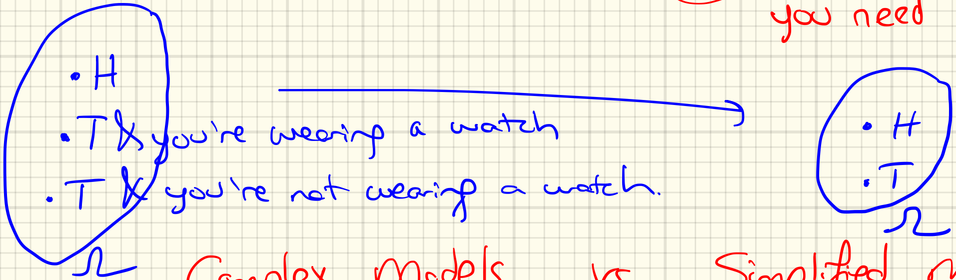
Sample Space (List of all outcomes) should be: From the experiment

① Mutually Exclusive (≡ Disjoint)
we get only one of the outcomes

② Collectively Exhaustive : all the outcomes that may happen in the experiment are in Ω.

<u>Ex:</u> Flip a coin ; you believe wearing a watch affects the outcome :

③ How much "<u>granularity</u>" you need in defining Ω



Complex Models vs Simplified Models

needed granularity

Einstein's (Occan's Razor) :
"Everything should be made as simple as possible, but no simpler".

→ Picking "right" amount of granularity is an art of engineering

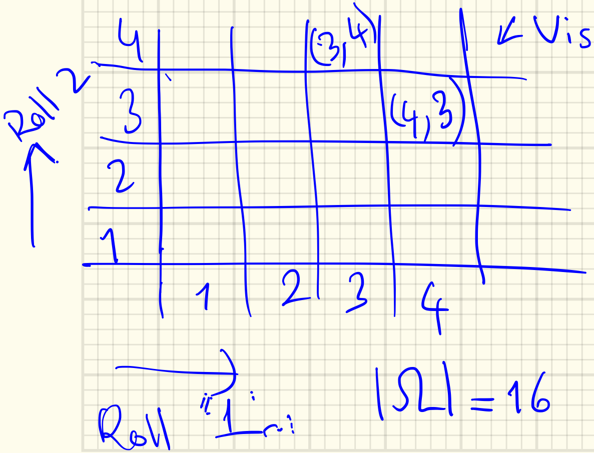
Ex: Rolling the dice twice : Single experiment, sequentially executed experiment.

Hypothetical dice w/ 4 faces

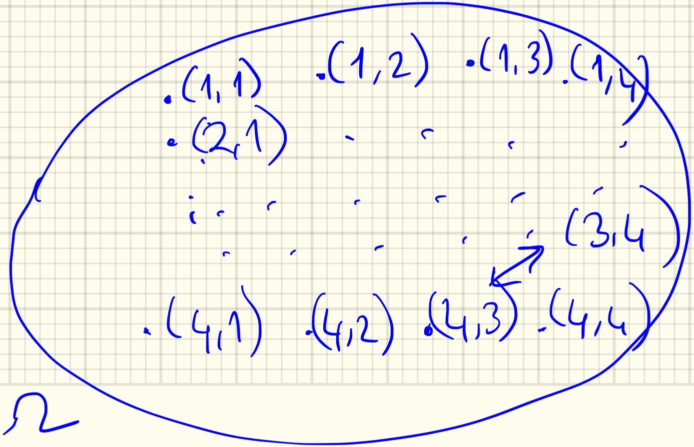
Single Roll outcomes : {1}, {2}, {3}, {4}

Q. What is the sample space of this experiment?

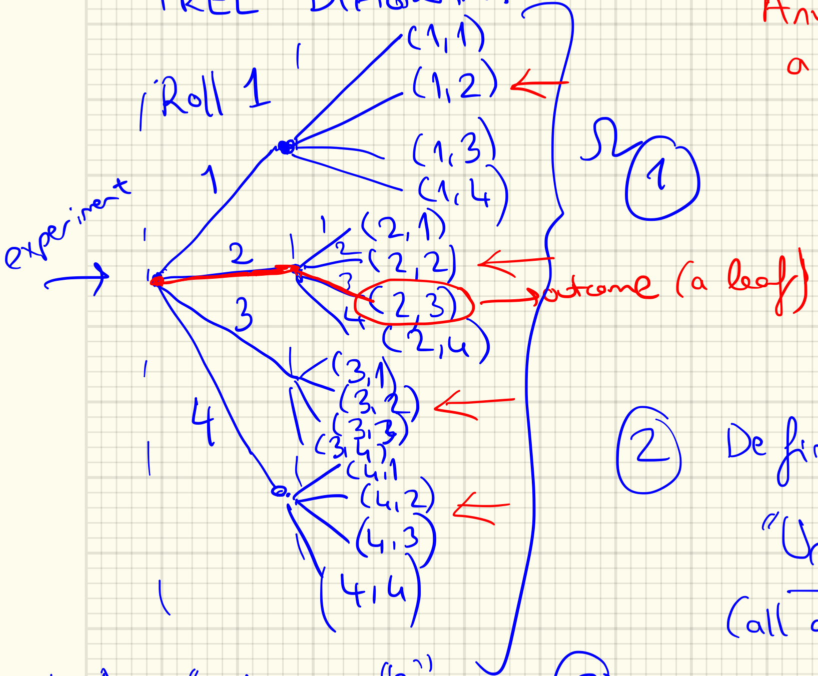
↙ Visualization of the sample space



$|\Omega| = 16$



For a sequential description of the experiment, we can use a TREE DIAGRAM



Any path in the tree is associated to a particular outcome & vice versa

leaves = 16 leaves

$|\Omega| = 16$

$\boxed{\text{Finite}}$ Sample Space

② Define probability law:

"Uniform" probability law:
(all outcomes are equally likely)

③ Event $E = \{(2,3)\}$

④ $P(E) = ? = \frac{|E|}{|\Omega|} = \frac{1}{16}$

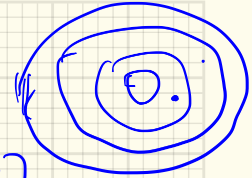
ex: A event = "getting a "2" in the 2nd roll".

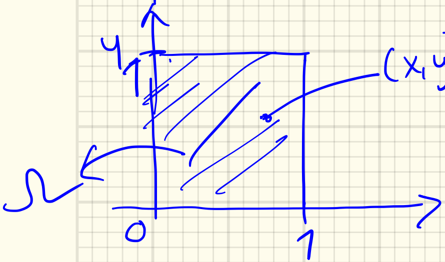
$|A| = 4$

$P(A) = \frac{4}{16} = \frac{1}{4}$

Infinite Sample Space (Continuous example)

Random Experiment : Throwing a dart into the square $[0,1]\times[0,1]$





(x,y) : a possible outcome $(x,y)\in[0,1]\times[0,1]$.

Q. Which outcome is more likely to occur?

$$P\big((x,y)=(0.7,0.6)\big) = \, ? \quad 0\,!$$

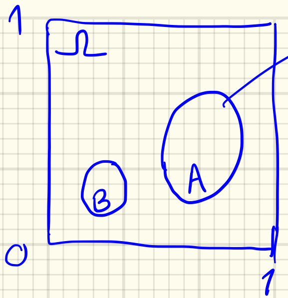
Any individual point (outcome) has zero probability

b/c $\exists$ (there exists) ∞-ly many real numbers

Q → How do we work w/ this?

A: We assign probabilities to SUBSETS of the sample space.

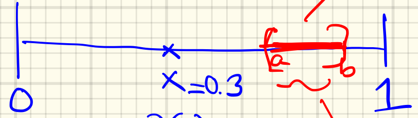
Subsets of Ω are called EVENTS



Event A : We assign a probability to event A.

$P(A) = \underline{\text{Area}} \text{ of } A$

1D Dart

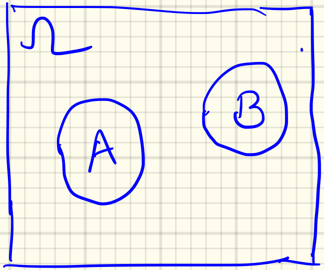


$P(x) = 0$!

$\text{Event } C = \{\overset{\text{dart}}{\text{hitting}} \text{ the } (a,b) \text{ interval}\}$

$\underline{P(C) = \text{length}(a,b)}$

Probabilities ∝ lengths of the intervals in 1D.



$P(A \cup B)$ = Total probability is the sum of individual probabilities of the two events A & B

→ Total Mass = Sum of the 2 masses.
Analogy

How should we make probability assignments ?

→ 1) want probabilities to be between [0, 1] →

b/c (because)
prob = zero → smt. (event/outcome) is not going to happen
prob = ONE → we're certain that smt is going to happen.

→ Want to satisfy other rules → all are summarized in

AXIOMS of PROBABILITY : all legitimate prob. models should obey

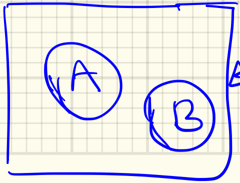
1) $P(A) \geq 0$ (nonnegativity)

2) $P(\Omega) = 1$ (normalization)

3) If $\underbrace{A \cap B = \emptyset}$ then $P(A \cup B) = P(A) + P(B)$

2) is b/c Ω is collectively exhaustive

3)



Union event : $A \cup B$: event that A occured or B occured,

Prob is the sum of their probabilities when they are mutually exclusive.

Refresh SETS : $A = \{ \xi_1, \xi_2, \ldots, \xi_n \}$ ← elements of A

collections of objects

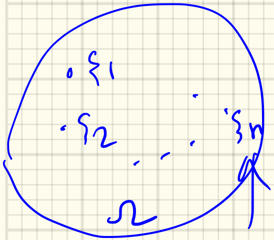
$\xi_i \in A$

$\xi_i \notin A$

- $\underline{\emptyset}$: empty set (impossible $\underline{\text{event}}$)
- If A is an event → then A^c (its complement) has to be an event.

Universal Set

Ω : (sample space) is an event : Certain Event

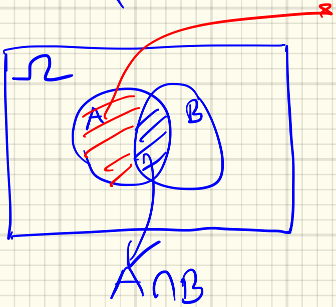


$\Omega^c = \emptyset$ is an event.

If a set has n elements, ∃ 2^n subsets including $\emptyset$ & itself.

Set Operations :
- Union : $A \cup B = \{x : x \in A$ or $x \in B\}$
  and
- Intersection : $A \cap B$
- Complement $A^c = \{ x : x \in \Omega$ but $x \notin A \}$
- Difference : $\underline{A - B} = \{x : x \in A$ and $x \notin B)$

$A - B = \underline{A \cap B^c}$

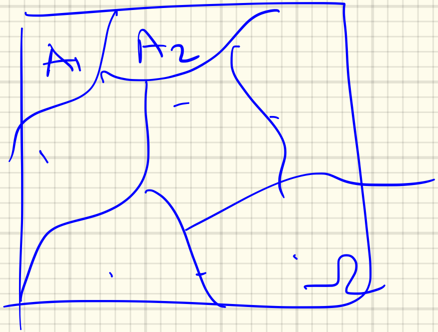


Def: Sets A & B are disjoint iff $A \cap B = \emptyset$

* A_i are called a <u>PARTITION of Ω</u> iff

1) A_i's are disjoint

2) $\bigcup_{i=1}^{\infty} A_i = \Omega$



For any subsets of Ω

- **Commutative Law:** $A \cup B = B \cup A$; $A \cap B = B \cap A$
- **Associative Law:** $A \cup \underbrace{(B \cup C)}_{} = (A \cup B) \cup C$

  $A \cap (B \cap C) = (A \cap B) \cap C$
- Distributive Law: $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- De Morgan Law: $\left(\bigcup_i A_i\right)^c = \bigcap_i A_i^c$

  $\left(\bigcap_i A_i\right)^c = \bigcup_i A_i^c$
- $A \subseteq B$: A is a subset of B

  $= B \supseteq A$

$\Rightarrow$ If $B \subseteq A$ & $A \subseteq B$ then $A = B$.

ex:

$[A \cap (B \cup C)]^c = ?$

$= A^c \cup (B^c \cap C^c)$

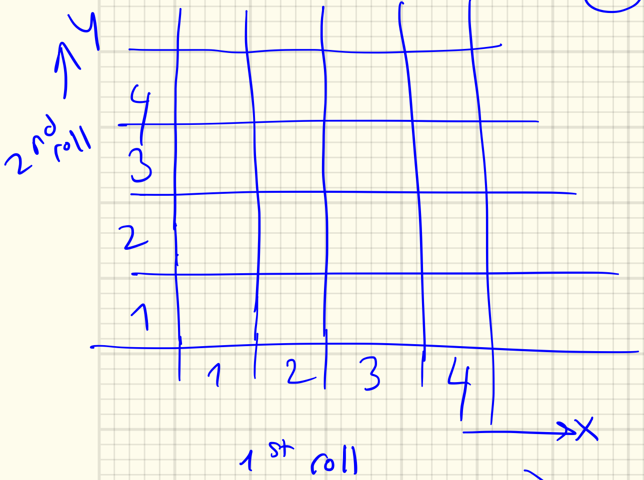
AXIOMS of PROBABILITY: 1) $P(A) \geq 0$ ←

2) $P(\Omega) = 1$

3) If $A \cap B = \emptyset \rightarrow P(A \cup B) = P(A) + P(B)$ ⇐

ex: Rolling two dice (w/ 4 faces)

1) Defined the sample space ✓



② Assign a probability law:

"Discrete" Uniform Probability law

Axiom ①

each outcome has a probability $= \frac{1}{16} \rightarrow P(A) \geq 0$ ✓

Q. Prob. that the 2nd roll gives "1" on the dice? $\boxed{A = \bigcup_i A_i}$

$P(\Omega)=1$

②

Axiom

$\rightarrow A_1 \quad A_2 \quad A_3 \quad A_4$

$(1,1) \quad (2,1) \quad (3,1) \quad (4,1)$

"singleton" event

$\boxed{\bigcap_i A_i = \emptyset}$

3rd axiom: $P(A) = P\left(\bigcup_i A_i\right) = \sum_i P(A_i) = 4 \cdot \frac{1}{16} = \frac{1}{4}$

Discrete Uniform Probability Law : →

some example
- fair coins
- dice
- well-shuffled decks.

All outcomes are equally likely :

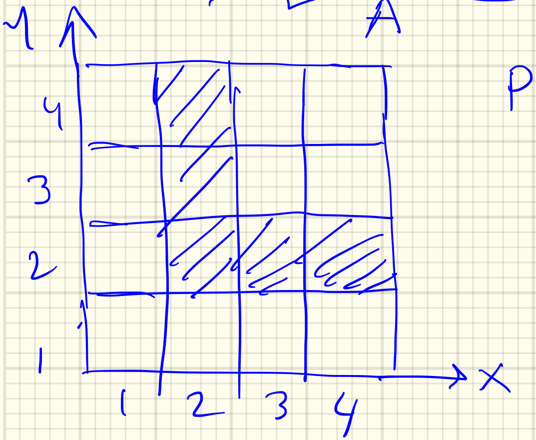
→ Computing probability → $\boxed{\text{Counting}}$ ↙

$$P(A) = \frac{|A|}{|\Omega|}$$

Q. $P(\underbrace{\min(X,Y) = 2}_{A}) = ?$

A

event : minimum of the 2 rolls is 2



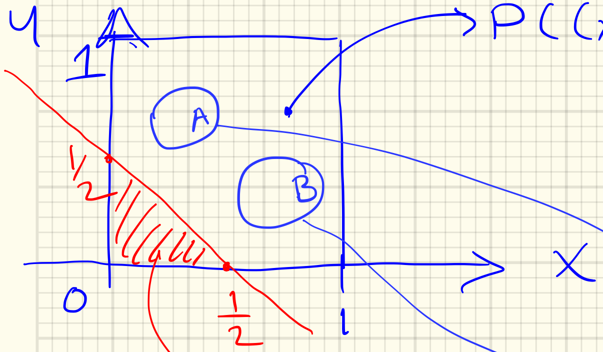
$$P(A) = \frac{5}{16}$$

Q. $P(X+Y \geq 5) = ?$ calculate.

Q. $P(X+Y \text{ is even}) = \frac{1}{2}$

Continuous Uniform Probability Law :

Recall the dart problem → we measured areas or lengths



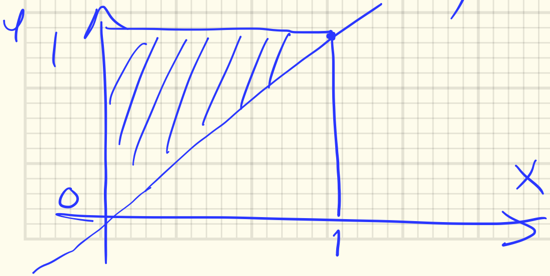
$$P((x,y) = (0.8, 0.6)) = 0$$

$$P(X + Y \le \frac{1}{2}) = \frac{1}{8}$$

If B has the double the area of A then

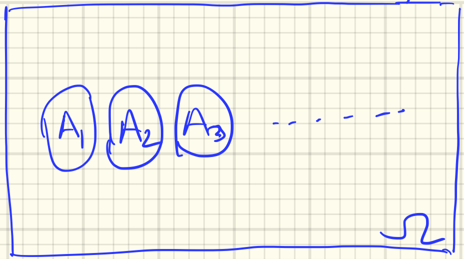
$$P(B) = 2 \times P(A)$$

$$P(X - Y \le 0) = ? \quad \frac{1}{2}$$



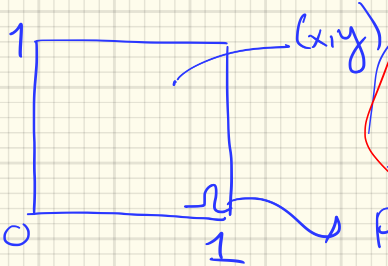
A_i are ∞-sequence of disjoint events ;

due Axiom 3_∞



$$P(A_1 \cup A_2 \cup A_3 \cdots\cdots) = \sum_{i=1}^{\infty} P(A_i)$$

$$= P(A_1) + P(A_2) + P(A_3) + \cdots\cdots$$



$$\bigcup \{(x,y)\}$$
$$\forall (x,y) \in [0,1] \times [0,1]$$

→ Uncountable set!

$$P(\Omega) = 1 = P\Big(\bigcup_{\forall x,y} \{(x,y)\}\Big)$$

$$\underline{1} \; ! \quad \cancel{=} \sum_{\forall x,y} \underbrace{P(\{(x,y)\})}_{0} = 0 \; !$$

→ Additivity Axiom → Countable Additive

3rd Axiom does not apply to uncountable sets.

$\bigcup_{i=1}^{\infty} A_i$;
countable seq of events!

$\{A_i\}_{i=1}^{\infty}$

countable set of events

$$\begin{array}{c} \{1,\ 2,\ 3,\ 4,\ \ldots\ldots\} \to \infty \\ \uparrow \quad \uparrow \quad \uparrow \\ \{A_1,\ A_2,\ A_3,\ \ldots\ldots\} \to \infty \end{array}$$

1-1 correspondence w/ +ve integers $\mathbb{Z}^+$.

Summarize: Probabilistic modeling : Random experiment is performed

1) Specify Sample Space :
   - → mutually exclusive
   - → collectively exhaustive
   - → "right" amount of granularity
2) Define a probability law :
3) Identify an Event of Interest :
4) Calculate probability of the event

+ (Kolmogorov's) Axioms of Probability

1) $\boxed{P(\Omega) = 1}$

2) $P(A) \geq 0$

3) $\{A_i\}_{i=}^{\infty}$ disjoint → $P(\bigcup_i A_i) = \sum_{i=1}^{\infty} P(A_i)$