YZV 231E

26.09.2022

Probability Theory & Stats

Week 2

Gü.

Recap: Essential elements in a probability model: Random experiment is performed:

1) Sample Space defined : Ω ← All outcomes of the experiment

2) Define a probability law :

3) Define an event(s)

4) Calculate probability of events

eg. 1 coin toss

$\{ \cdot H, \cdot T \}$

$\Omega = \{H, T\}$

$|\Omega| = 2$

Experiment : Verbal description

eg. Consider random experiment of 4 coin tosses.

$\Omega = \{H,T\} \times \{H,T\} \times \{H,T\} \times \{H,T\} \rightarrow |\Omega| = 2^4 = 16$

1) $\Omega = \Big\{ \{HHHH\}, \{HHHT\}, \{HHTT\}, \dots, \{TTTT\} \Big\}$

2) All outcomes are equally likely $\equiv$ Discrete Uniform Prob Law

3) $P(\underbrace{\{THTH\}}_{E}) = \frac{1}{16}$ 4

$E = \{\text{getting exactly 3 Heads}\} = \{\{HHHT\}, \{HHTH\}, \{HTHH\}, \{THHH\}\}$

$P(E) = \frac{|E|}{|\Omega|} = \frac{4}{16} = \frac{1}{4}$

Recap: Our probability model should obey 3 axioms of probability.

Def: A collection of subsets of Ω is called a σ-field (σ-algebra) or Borel field $\mathcal{B}$, if it satisfies:

i) $\emptyset \in \mathcal{B}$

ii) If $A \in \mathcal{B}$ then $A^c \in \mathcal{B}$ ($\mathcal{B}$ is closed under complementation)

iii) If $A_1, A_2, \ldots \in \mathcal{B}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{B}$ ($\mathcal{B}$ is closed under countable unions)

Note: Due De Morgan's law, $\mathcal{B}$ is also closed under countable intersections.
($\mathcal{B}$ is closed under countably ∞ # of set operations)

* If Ω is finite or countable
then $\mathcal{B} = \{$ all subsets of Ω, including Ω itself $\}$

ex. Let $\Omega = \{1, 2, 3\} \rightarrow \mathcal{B} = \{\{1,2,3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1\}, \{2\}, \{3\}, \emptyset\}$
$|\mathcal{B}| = 2^3 = 8$
largest $\mathcal{B}$ field

Another $\mathcal{B} = \{\emptyset, \{1,2,3\}\}$: smallest $\mathcal{B}$-field

Q: $\mathcal{B} = \{\{1,2,3\}, \emptyset, \{1,2\}, \{3\}\}$
Is this a Borel field? yes

* let $\Omega = \{-\infty, \infty\}$ the real line.

A_i : $[a,b]$, (a,b), $(a,b]$, $[a,b)$ intervals in $\mathbb{R}$ $\forall a,b \in \mathbb{R}$.

Choose $\mathbb{B}$ to contain all sets of the form of A_i.

$\mathbb{B}$ contains all sets that can be formed by taking countably ∞ unions & complementations of such sets

ex: $\underbrace{A_i}_{} = \frac{1}{2^i}$, $i = 1, ..., \infty$ Q: is this a countably ∞ set? yes.

can be put to 1-1 corresp. w/ set of natural numbers.

→ Events are certain (possibly all) subsets of Ω forming a Borel-field $\mathbb{B}$.

Axioms of Probability: Given a sample space Ω, and an associated σ-field $\mathbb{B}$, a probability fn. P w/ domain $\mathbb{B}$, that satisfies:

$P : \mathbb{B} \longrightarrow \cancel{\mathbb{R}^+}$ $[0,1]$

(1) $P(A) \geq 0$, for all $A \in \mathbb{B}$

(2) $P(\Omega) = 1$

(3) If $A_1, A_2, ... \in \mathbb{B}$ are pairwise disjoint ($A_i \cap A_j = \emptyset$ $\forall i,j$), then

$\bigcup_i A_i \in \mathbb{B} \rightarrow P(\bigcup_i A_i) = \sum_{i=1}^{\infty} P(A_i)$.

Only 3 axioms !! We can derive many others based on these:

Thm: If P is a probability fn. (i.e. it satisfies the 3 axioms of prob), and $A \in \mathcal{B}$, then:

$$\begin{cases} 1)\ P(A) \geq 0, & \forall A \in \mathcal{B} \\ 2)\ P(\Omega) = 1 & \\ 3)\ P(\bigcup_i A_i) = \sum_i P(A_i) & A_i, A_j \text{ disjoint} \end{cases}$$

* $P(\emptyset) = 0$

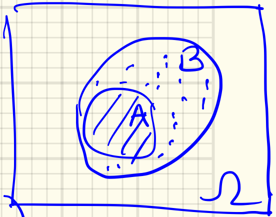
* $P(A) \leq 1$; $A \subset \Omega$

$$\Omega = A \cup A^c$$

$$P(\Omega) = 1 = P(A) + \underbrace{P(A^c)}_{\geq 0} \Rightarrow P(A) \leq 1$$

* $P(A^c) = 1 - P(A)$

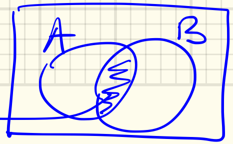
* If $A \subset B$ then $P(A) \leq P(B)$



$$B = A \cup (A^c \cap B)$$ disjoint

$$P(B) = P(A) + \underbrace{P(A^c \cap B)}_{\geq 0} \longrightarrow P(A) \leq P(B)$$

* $P(B \cap A^c) = P(B) - P(A \cap B)$



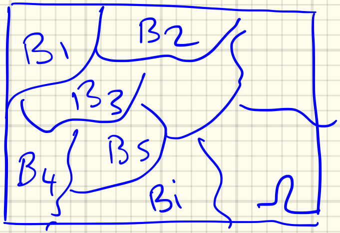
* $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

exercise: show this.

* $\{B_i\}$ be any partition of Ω

Def: (Partition of Ω): $B_i \cap B_j = \emptyset \quad \forall i, j$ &

$$\bigcup_i B_i = \Omega$$



$$P(A) = \sum_{i=1}^{\infty} P(A \cap B_i)$$

Pf: $\Omega = \bigcup_{i=1}^{\infty} B_i$, $A = A \cap \Omega = A \cap \left(\bigcup_i B_i\right)$

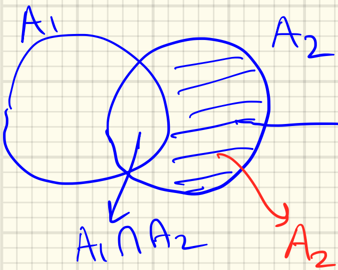
$$A = \bigcup_i (A \cap B_i)$$

$$P(A) = P\left(\bigcup_i (A \cap B_i)\right) = \sum_{i=1}^{\infty} P(A \cap B_i)$$

* Union Bound : $P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i)$

Note that A_i's are <u>not</u> necessarily disjoint. (finite or countable)

Show for 2 sets



$$A_1 \cup A_2 = A_1 \cup \overbrace{(A_2 \cap A_1^c)}^{\text{disjoint}}$$

$A_2 \cap A_1^c \subset A_2$

$$P(A_1 \cup A_2) = P(A_1) + P(\underbrace{A_2 \cap A_1^c}_{\subset A_2})$$

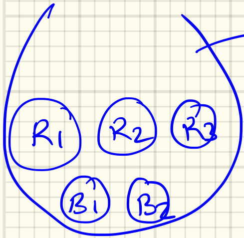
$$P(A_1 \cup A_2) \leq P(A_1) + P(A_2)$$

(we showed for 2 sets)

This generalizes to countably ∞-sequences A_i.

exercise : Show these derived axioms.

Ex: Urns (torba) $\rightarrow$ (Sampling):



$\rightarrow$ 3 Red Balls
2 Black Balls

We sample a ball $\rightarrow$ we replace it back.

List of Balls
R_1, R_2, R_3, B_1, B_2

$E = \{$ getting first a "Red" then a "Black" ball $\}$

$P(\text{"}R, B\text{"}) = ?$

$\{R, B\}$

Experiment: Sample 2 balls sequentially, independently.

sample a ball

$\Omega_1 = \{R_1, R_2, R_3, B_1, B_2\}$

$\Omega_2 = \{R_1, R_2, R_3, B_1, B_2\}$

$\Omega = \Omega_1 \times \Omega_2$

Ω $\downarrow$ sample space ; Cartesian product

1st Ball Draw $\times$ 2nd ball draw

$E = \{\{R_1, B_1\}, \{R_1, B_2\}, \{R_2, B_1\}, \{R_2, B_2\}, \{R_3, B_1\}, \{R_3, B_2\}\}$

$P(E) = \dfrac{|E|}{|\Omega|} = \dfrac{6}{25}$ $\leftarrow$ Here we manually counted cardinality of the events ✓

(Discrete uniform prob. law)

→ Now, deal a 52-card deck.

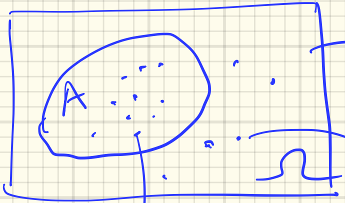
$$P(\underbrace{\text{5 cards you draw contain 4 aces}}_{E}) = ?$$

Sample space size ? → Huge space !

→ That's why we study $\boxed{\text{COUNTING}}$: → part of the COMBINATORICS

↳ huge field.

probabilistic modeling.



Ω: finite sample space

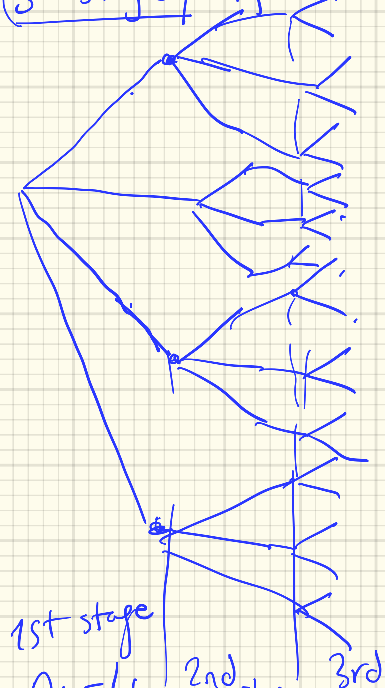
say N outcomes ; equally likely

$\frac{1}{N}$ likelihood / probability for each outcome

$$P(A) = \frac{|A|}{|\Omega|}$$

Sequential experiments → counting

say $\boxed{3 \text{ stage}}$ experiment



1st stage

$n_1 = 4$

2nd stage

$n_2 = 3$

3rd stage

$n_3 = 2$ outcomes/experiment

How many outcomes in total

$$n_1 \times n_2 \times n_3 = 24.$$

Ex: # Licence plates w/ 3 letters 4 numbers

34

$22 \times 22 \times 22 \quad 10 \times 10 \times 10 \times 10 = (22)^3 (10)^4 = |\Omega|$

22 letters (from the Turkish alphabet)

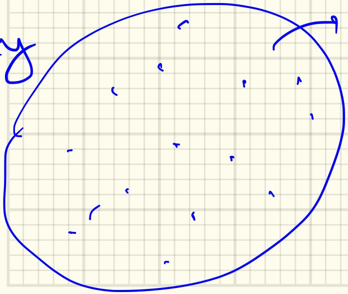
w/ replacement $\equiv$ repetition allowed.

→ If repetition is not allowed

w/o replacement

$22 \times 21 \times 20 \quad 10 \times 9 \times 8 \times 7 = |\Omega|$

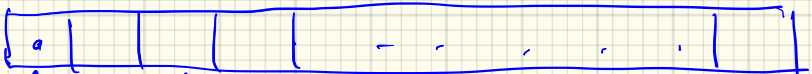
say



n element set

→ choose k-elements in an ordered fashion

k boxes



$n \quad n-1 \quad n-2 \quad \ldots \quad n-(k-1)$

$n-k+1$

Permutation: # possible arrangements of n objects:

$$n(n-1)(n-2)\cdots 1 = n!$$

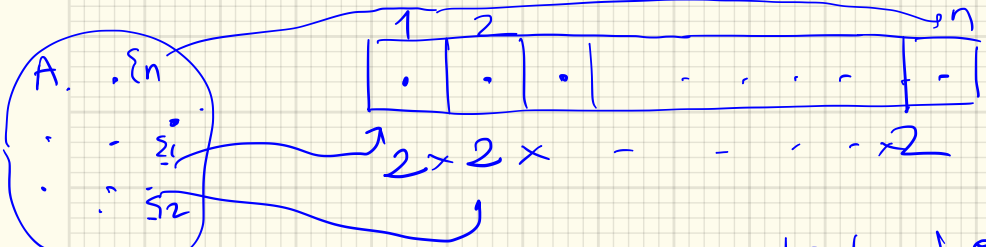
= n factorial ways to order n objects

k-permutation: $\dfrac{n!}{(n-k)!} = n(n-1)\cdots(n-k+1) \triangleq n_k$

$\downarrow$

Selecting k objects from a collection of n objects.

Ex: We have a set $A = \{1, 2, 3, \ldots, n\}$



$$2 \times 2 \times \cdots \times 2 = 2^n$$

Q. How many subsets does A have?

$A = \{1, 2\} \longrightarrow$ verify

Def: Number of subsets of an n-element set $= 2^n$.

Ex: Urn: Five balls numbered $1, 2, 3, 4, 5$, are drawn from an urn w/o replacement.

What is the probability that they will be drawn in the same order as their number?

1	2	3	4	5

$\leftarrow E$.

$$|\Omega| = 5!$$

$$P(E) = \frac{1}{5!} = \frac{1}{120}.$$

Ex: We are given a fair die (w/ six side): roll it 5 times (independent)

- $P(\{5, 4, 2, 2, 1\}) = ?$ $\quad \frac{1}{6^5}$

$$|\Omega| = 6^5$$

w/ replacement.

$$6 \times 6 \times 6 \times 6 \times 6 = 6^5$$

- $A = \{$all rolls give different numbers$\}$

$$P(A) = \frac{6 \times 5 \times 4 \times 3 \times 2}{6^5}$$

w/o replacement

$$6 \times 5 \times 4 \times 3 \times 2 = 6_5 \; (= n_k)$$

$n_k = 6_5 = \frac{n!}{(n-k)!} = \frac{6!}{1!} = 6!$

$0! = 1$

if $k = n$ $\quad \frac{6!}{0!} = 6!$

Next → Want to form k people committees

n people

choose k → k people

Choose k among n

$\binom{n}{k}$: n choose k

Start w/ ordered subsets

$n \quad (n-1) \quad (n-2) \quad \cdots \quad n-k+1$

choices

$\underbrace{\qquad\qquad\qquad}_{k \text{ elements}}$

$\frac{n!}{(n-k)!} = n_k$

Q: How many ways can I permute these k-elements?

A: $k!$

$\frac{n!}{(n-k)!} \cdot \frac{1}{k!}$: (unordered)

$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$ →

"n choose k"

$\binom{n}{k}$: (binomial coefficient) $\longrightarrow$ how to choose k-element subsets out of n-elements (Unordered)

Combination

If $k=n$ $\quad \binom{n}{n} = \frac{n!}{0!\,n!} = 1$ ✓

Note $\boxed{\sum_{k=0}^{n} \binom{n}{k} = 2^n}$ = Total # subsets of an n-element set

sums of all possible # element subsets of an n-element set

Ex: 52-card deal : prob of having 4 "1"s (aces) ? (E)

Choose 5 cards

$$|\Omega| = \binom{52}{5} \quad : \text{sample space size}$$

$$P(E) = \frac{48}{\binom{52}{5}} = \frac{48}{\frac{52!}{47!\,5!}}$$

$$\binom{4}{4} = 1 \times 48 \text{ ways to choose the 5th slot.}$$

- $P(\text{having 4 cards of the same kind}) = ?$

$$\frac{13 \times 48}{\binom{52}{5}}$$

→ 48

13

1	1	1	1
2	2	2	2
3	3	3	3
⋮			
10	10	10	10
11	11	11	11
12	12	12	12
13	13	13	13

Ex: A class consists of 30 students, 20 are freshmen
10 are sophomores

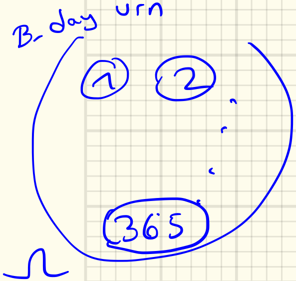
If 5 students are selected at random;
what is the prob. that they will all be sophomores?

S	S	S	S	S

$$\frac{\binom{10}{5} \rightarrow |E|}{|\Omega| \rightarrow \binom{30}{5}} = \frac{\frac{10!}{5!\,5!}}{\frac{30!}{25!\,5!}} = \frac{10.9.8.7.6}{30.29.28.27.26} = 0.018$$

Ex: Birthday problem: A class has N students.

$A = \{$ at least 2 students have the same bday $\}$



Sampling w/ replacement

1	2			-	-	-	-	-	.	N

$365 \times 365 \quad - \quad - \quad - \quad - \quad 365$

$\Rightarrow |\Omega| = 365^N \longrightarrow$ consider complement of A $\longrightarrow$

$A^c = \{$ no two students have the same b/day $\}$

1 2 ... N

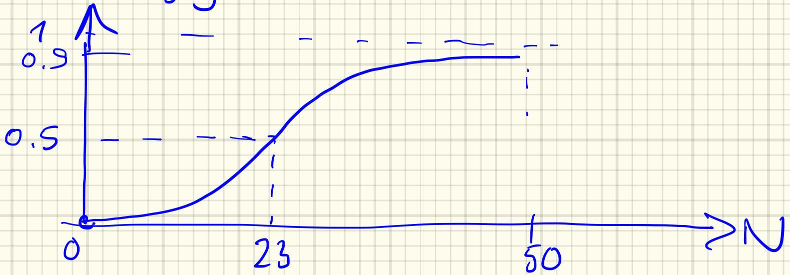
$365 \times 364 \times 363 \; - \; - \; - \; - \; - \; 365-N+1$

$\Rightarrow (365)_N = |A^c|$

$$P(A^c) = \frac{(365)_N}{365^N} \rightarrow P(A) = 1 - P(A^c)$$

$$= 1 - \frac{(365)_N}{365^N}$$

Ex 3.12 [S Kay]



Summarize: # possible arrangements of size k from N objects

	w/o replacement	w/ replacement
Ordered	$N_k = \frac{N!}{(N-k)!}$	N^k
Unordered	$\binom{N}{k}$	$\binom{N+k-1}{k}$ ← given w/o proof.

4th case: sampling w/ replacement unordered: $\{1, 2, 3\}$

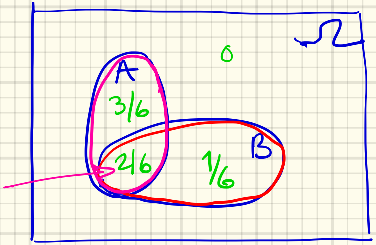
sample 2 balls:

$A = (1,1), (1,2), (1,3), (2,2), (2,3), (3,3) \Rightarrow \binom{4}{2}$

$|A| = 6$

CONDITIONAL PROBABILITY

: Revised beliefs ≡ probabilities w/ some new (partial) information



$P(A)$ ✓ → Now we have new info

Event B occured

$$\underbrace{P(A \mid B)}_{\text{prob of A given that B occured}} = \frac{P(A \cap B)}{\underbrace{P(B)}} = \frac{2/6}{3/6} = \frac{2}{3}$$

$P(B) = \frac{3}{6}$

We know B occured

$$P(B \mid B) = 1 = \frac{P(B \cap B)}{P(B)} = 1$$

$$P(A \mid B) \triangleq \frac{P(A \cap B)}{P(B)}, \quad \text{undefined if } P(B) = 0.$$

$$\rightarrow \boxed{P(A \cap B) = P(A \mid B) \cdot P(B)}$$

Conditional probabilities are ordinary probabilities → satisfy Axioms of Prob

$$P(A \mid C) \geq 0$$

$$P(\Omega \mid C) = 1$$

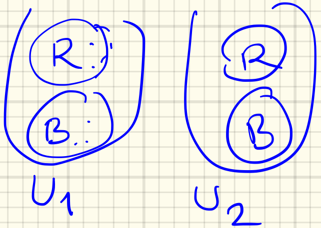
$$P(A \cup B \mid C) = P(A \mid C) + P(B \mid C)$$

3 axioms are satisfied by conditional prob.

A & B are disjoint

Ex: Compound experiment : Two urns.

We select one of the urns randomly



→ $\boxed{U_1}$: proportion of red balls : p_1 ⇐ prob of red balls

" black " $(1-p_1)$

$\boxed{U_2 :}$ " red " : p_2

" black " $(1-p_2)$

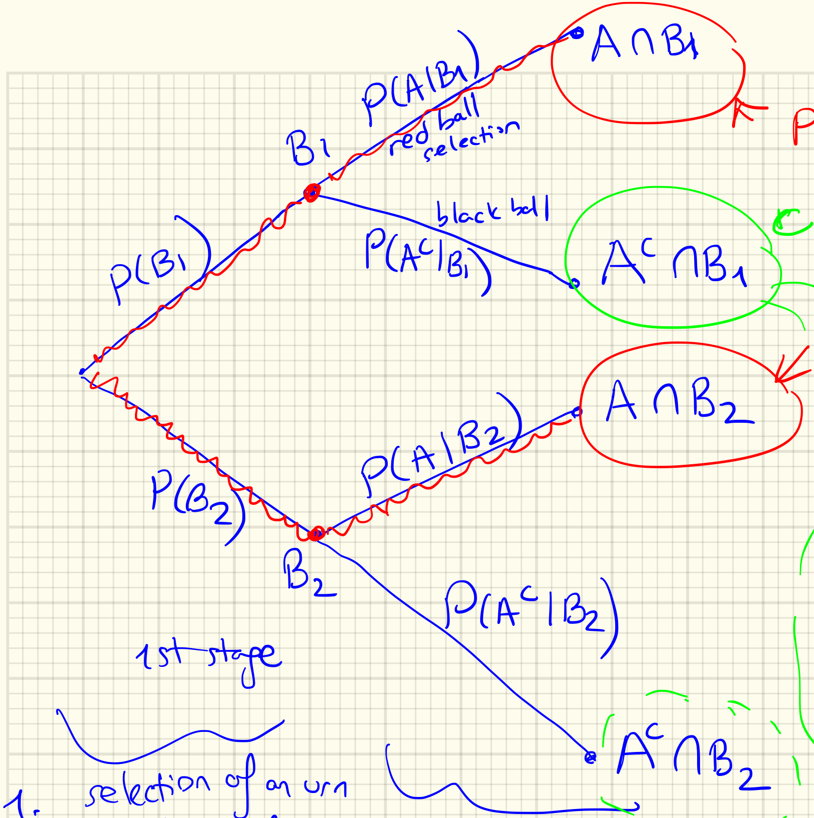
$A = \{\text{selecting a red ball}\} \Rightarrow P(A) = ?$

$B_1 = \{U_1 \text{ is selected}\}$

$B_2 = \{U_2 \text{ is selected}\}$

2-stage experiment

use a tree diagram ⟹



1. selection of an urn equally likely

2. drawing a ball w/ equally likely probabilities

$$P(A) = \underbrace{P(A\cap B_1)} + P(A\cap B_2)$$

$$= P(B_1)\cdot P(A|B_1) + P(B_2)\cdot P(A|B_2)$$

$$= \frac{1}{2}\cdot p_1 + \frac{1}{2}\cdot p_2$$

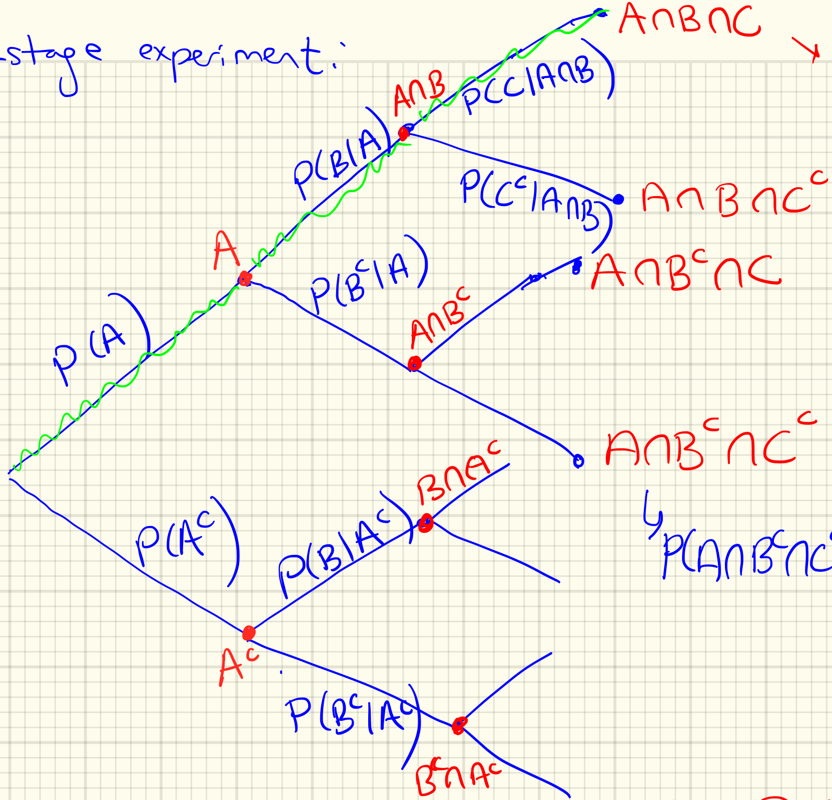
exercise:

$\Rightarrow P($selecting a black ball$)$ $E =$

$\Rightarrow P(C =$ selecting a black ball from U_1 urn 1$)$

$P(E)$

3-stage experiment:



$P(A\cap B\cap C)$: move along the tree

multiply the probabilities along the path

$P(A\cap B\cap C) = p(A)\cdot p(B|A)\, P(C|A\cap B)$

$P(A\cap B^c\cap C^c) = P(A)\,P(B^c|A)\,P(C^c|A\cap B^c)$

Generalize: Multiplication Rule (= Probability Chain Rule)

$$P\left(\bigcap_i A_i\right) = p(A_1)\cdot p(A_2|A_1)\, p(A_3|A_1\cap A_2)\cdots p\left(A_n \,\middle|\, \bigcap_{i=1}^{n-1} A_i\right)$$

Example: 3 cards are drawn from a 52-card deck w/o replacement.

P(none of the 3 cards is a heart) = ?

13 Hearts

Define $A_i = \{ i^{th}$ card is not a heart $\}$, $i = 1, 2, 3$.

we want $P(A_1 \cap A_2 \cap A_3) = ?$

$$= P(A_1)\, P(A_2 \mid A_1) \cdot P(A_3 \mid A_1 \cap A_2)$$

$$P(A_1) = \frac{39}{52} \longrightarrow 39 \text{ cards that are not hearts in the deck.}$$

$$P(A_2 \mid A_1) = \frac{38}{51} \quad , \quad P(A_3 \mid A_1 \cap A_2) = \frac{37}{50}$$

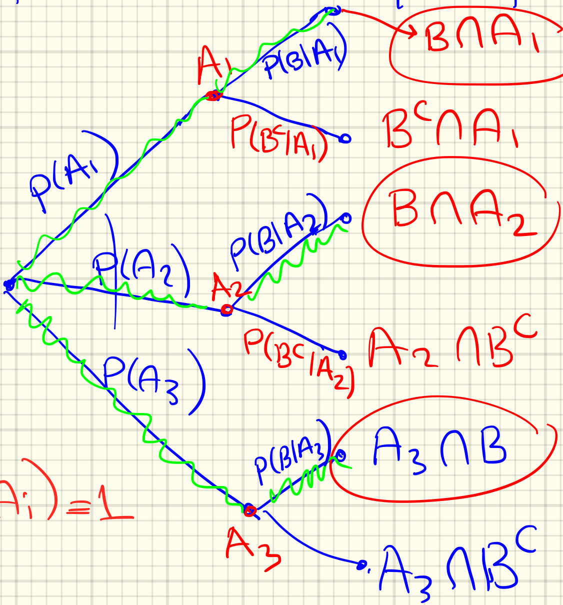
$$P(A_1 \cap A_2 \cap A_3) = \frac{39 \cdot 38 \cdot 37}{52 \cdot 51 \cdot 50}$$

By counting

$$\longrightarrow \frac{\binom{39}{3}}{\binom{52}{3}} = \checkmark$$

Total Probability Theorem:

Partition the sample space into $\bigcup_i A_i$'s, A_i's disjoint

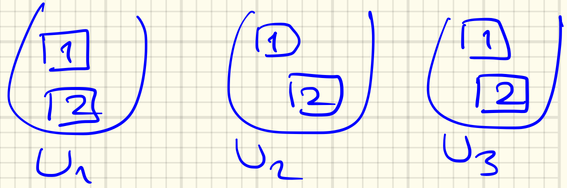


(A_1 happened, B happened)

(A_1 happened, B not happened)

$$\sum_{i=1}^{\infty} P(A_i) = 1$$

Again, like sampling from 3 urns, Urns have 2 types of products



B = selecting product 1.

A_1 : select urn 1

A_2 : " 2

A_3 : " 3

Total prob of B

$$P(B) = \underbrace{P(A_1)\,P(B|A_1)} + \underbrace{P(A_2)\cdot P(B|A_2)} + \underbrace{P(A_3)\,P(B|A_3)}$$

3 ways B can happen.

∴ Total prob. is weighted addition of conditional probabilities